

**SMC Department of Mathematics**  
**Problem of the Week**

Week 1 Solution

**The Two Cubes:** Two cubes with integral side length have their combined volume equal to the combined length of their edges. What are the dimensions of the cubes?

**Solution:** Let  $x$  and  $y$  be the side lengths of the two cubes. Each cube has twelve edges, so  $x$  and  $y$  satisfy the equation  $x^3 + y^3 = 12x + 12y$ , or  $x^3 + y^3 = 12(x + y)$ . The sum  $x + y > 0$ , so dividing both sides by  $x + y$  we find that  $x^2 - xy + y^2 = 12$ . We can think of this equation as a quadratic, and solve for  $x$  in terms of  $y$ . Rearranging the terms, we get  $x^2 + xy + y^2 - 12 = 0$ . Applying the quadratic formula with  $a = 1$ ,  $b = -y$ , and  $c = y^2 - 12$ ,  $x = \frac{y \pm \sqrt{y^2 - 4(y^2 - 12)}}{2} = \frac{y \pm \sqrt{48 - 3y^2}}{2}$ .

This tells us two things. First,  $48 - 3y^2 \geq 0$ . Thus  $0 < y \leq 4$ . Second, for  $x$  to be an integer,  $\sqrt{48 - 3y^2}$  must be a whole number. Checking  $y = 1, 2, 3, 4$ , we see that either  $y = 2$  and  $x = 4$ , or  $y = 4$  and  $x = 2$ . Thus one cube has side length two, the other four.