Poisson Geometry

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Abstract

This paper is a survey of Poisson geometry, with an emphasis on global questions and the theory of Poisson Lie groups and groupoids.

1 Introduction

Poisson [105] invented his brackets as a tool for classical dynamics. Jacobi [63] realized the importance of these brackets and elucidated their algebraic properties, and Lie [82] began the study of their geometry. After a long dormancy, Poisson geometry has become an active field of research during the past 30 years or so, stimulated by connections with a number of areas, including harmonic analysis on Lie groups (Berezin [11]), infinite dimensional Lie algebras (Kirillov [67]), mechanics of particles and continua (Arnold [5], Lichnerowicz [80], Marsden–Weinstein [96]), singularity theory (Varchenko–Givental' [122]), and completely integrable systems (Gel'fand–Dikii [45], Kostant [75]), to mention just a few examples.

A *Poisson algebra* is a commutative associative algebra $\mathcal{A}$ over $\mathbb{R}$ carrying a Lie algebra bracket $\{\ ,\ \}$ for which each adjoint operator $X_h = \{\ , h\}$ is a derivation of the associative algebra structure. There are also *noncommutative* Poisson algebras [15, 40, 124, 139], but we will not treat them in this paper. Of course, one can replace $\mathbb{R}$ by another field.

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For most of this paper, $\mathcal{A}$ will be the algebra $C^\infty(M)$ of smooth functions on a manifold M, in which case the bracket is called a *Poisson structure* on M, and $(M, \{\ ,\ \})$ is called a *Poisson manifold.* The derivations X_h are represented in this "spatial" picture by vector fields, which are called *hamiltonian* vector fields.

A *Poisson morphism* $\mathcal{A}_1 \to \mathcal{A}_2$ is a map between Poisson algebras which is a homomorphism for both the associative and Lie algebra structures. When $\mathcal{A}_1 = C^\infty(M_1)$ and $\mathcal{A}_2 = C^\infty(M_2)$, the algebra morphism is the pullback operation for a smooth map $M_1 \leftarrow M_2$ which preserves Poisson brackets. We call these maps *Poisson maps.* (See [4] or [14] for a proof that any associative algebra homomorphism between algebras of smooth functions is the operation of pullback by some smooth map.)

Poisson manifolds occur as phase spaces for classical particles (and, in the infinite dimensional case, fields), but Poisson geometry is also relevant to the algebras of observables in quantum mechanics, as well as to more general noncommutative algebras. In fact, Kontsevich [69] has just shown that the classification of formal deformations of the algebra $C^\infty(M)$ for any manifold M is equivalent to the classification of formal families of Poisson structures on M. Furthermore, the derivations X_h of a Poisson algebra behave rather like the inner derivations of a noncommutative algebra, and consequently there are strong analogies between Poisson geometry and noncommutative algebra. We do not discuss these analogies here, but refer the reader to [18, 114].

Another category with close relations to Poisson geometry is that of Lie algebroids, while the corresponding global objects, Lie groupoids, can be used to encode the automorphisms of Poisson manifolds generated by the hamiltonian vector fields X_h. It is no accident that groupoids have also been a rich source of noncommutative algebras [23, 132]. We discuss Lie algebroids and groupoids in Section 7.

There are now several books available with extensive discussions of Poisson geometry [13, 18, 54, 65, 79, 120]. The reader is referred to them for further details of much of what is merely sketched here. This paper, then, is an idiosyncratic survey of Poisson geometry, marked by my own interests, and those of people with whom I have had close personal contact, including several of my former students. I have tried to make the bibliography representative of the literature, but it is far from complete. I apologize for all the relevant citations which have been omitted.

2 Poisson manifolds and maps

Since the bracket $\{f, g\}$ of functions on a Poisson manifold M is a derivation in each argument, it depends only on the first derivatives of f and g, and hence it can be written in the form

$$\{f, g\} = \pi(df, dg)$$

where $\pi \in \Gamma(\wedge^2 TM)$ is a field of skew-symmetric bilinear forms on T^*M, i.e. a *bivector field*. We call π the *Poisson tensor*. The Jacobi identity for the bracket implies that π satisfies an integrability condition which is a quadratic first-order (semilinear) partial differential equation in local coordinates, and has the invariant form $[\pi, \pi] = 0$, where the bracket here is the Schouten–Nijenhuis bracket on multivector fields (see [120]).

In addition to the tensor π, we will occasionally refer to the associated bundle map $\tilde{\pi} : T^*M \to TM$ defined by $a(\tilde{\pi}(b)) = \pi(b, a)$ for cotangent vectors a and b.

Poisson maps can be characterized by the following property of their graphs. A submanifold C of a Poisson manifold M is called *coisotropic* if the set of functions on M which vanish on C is closed under Poisson bracket. A differentiable map ϕ between Poisson manifolds M and N is a Poisson map if and only if its graph is a coisotropic submanifold of $\overline{M} \times N$, where $\overline{M}$ is M with its Poisson structure multiplied by -1. With this fact in mind, we call any coisotropic submanifold of $\overline{M} \times N$ a *Poisson relation* from M to N. The *coisotropic calculus* [131] in Poisson geometry is based on the fact that the composition of two Poisson relations is again a Poisson relation, as long as the composition satisfies a "clean intersection" assumption which guarantees that this composition is a smooth manifold. This calculus extends to Poisson geometry the lagrangian calculus (composition of canonical relations) [126] for symplectic manifolds.

3 Local structure of Poisson manifolds

Local Poisson geometry begins with the following theorem.

Theorem 3.1 (Splitting Theorem [127]) *Centered at any point O in a Poisson manifold M, there are coordinates* $(q_1, \ldots, q_k, p_1, \ldots, p_k, y_1, \ldots, y_\ell)$

such that

$$\pi = \sum_{i=1}^{k} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{i,j=1}^{\ell} \varphi_{ij}(y) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \qquad \textit{and} \qquad \varphi_{ij}(0) = 0 \ .$$

When $\ell = 0$, the Poisson structure is called *symplectic*, and the theorem is Darboux's theorem. Hence, a general Poisson manifold is isomorphic near each point to the product of an open subset of a standard symplectic manifold $\mathbb{R}^{2k}$ and a Poisson manifold for which the Poisson tensor vanishes at the point in question. The local classification question can thus be reduced to that for structures of the second type, which we call *totally degenerate*.

3.1 Symplectic leaves

The symplectic and totally degenerate factors are quite different when they are viewed intrinsically. The symplectic factor is an open subset of a well-defined submanifold $\mathcal{O}_O$, called the *symplectic leaf* through O. M is a disjoint union of these symplectic leaves, and the Poisson bracket on M "assembles" the canonical Poisson brackets on these leaves. On an open dense subset of M, the *regular* part, the dimension of the leaves is locally constant, and they form a foliation.

The totally degenerate factor, on the other hand, lives more naturally on a "quotient" of a neighborhood of O by a foliation (not intrinsically defined!) having $\mathcal{O}_O$ as a leaf. It can be shown that the degenerate factor is well-defined, but only up to isomorphism; the isomorphism class of this *transverse structure* is the same for all points of the symplectic leaf $\mathcal{O}_O$.

3.2 Totally degenerate structures and linearization

Since our study will be local, we can assume that the underlying manifold is a vector space V, and that the point of total degeneracy is the origin. In linear coordinates, the coefficients of the Poisson tensor can then be written as linear functions plus higher order terms. If the higher order terms all vanish, the dual space V^* of linear functions on V is a Lie algebra $\mathfrak{g}$, so that $V = \mathfrak{g}^*$. The Poisson structure is completely determined by the Lie algebra structure on $\mathfrak{g}$: in linear coordinates $(x_1, \ldots . x_n)$ on $\mathfrak{g}^*$ (which can also be viewed as a basis of $\mathfrak{g}$), the Poisson tensor is

$$\pi = \frac{1}{2} \sum_{i,j,k=1}^{n} c_{ijk} x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

where c_{ijk} are the structure constants of the Lie algebra with respect to this basis. We call such Poisson structures on the dual spaces of Lie algebras *Lie–Poisson structures.*

A Poisson structure which becomes linear when expressed with respect to suitable local coordinates at a point O is said to be *linearizable* at O. The candidate for the corresponding Lie algebra structure is determined by the Poisson structure itself: at a point O where a Poisson structure on M vanishes, there is a well-defined bracket on the differentials at O of functions on M, defining the structure of a Lie algebra (the *tangent Lie algebra*) on the cotangent space $T^*_O M$ and hence a Lie–Poisson structure (the *tangent Poisson structure*) on the tangent space $T_O M$. The structure on M is linearizable at O iff it is locally isomorphic to its tangent Poisson structure at that point.

There are Lie algebras $\mathfrak{g}$ such that any totally degenerate Poisson structure with tangent Lie algebra $\mathfrak{g}$ is linearizable. These algebras are analogous to nondegenerate quadratic functions, from which additional higher order terms can be removed by a coordinate transformation. For this reason, we have called such Lie algebras *nondegenerate* in [127]. However, the juxtaposition of "degenerate" and "nondegenerate" in this context seems a bit confusing, so we will follow the terminology of singularity theory and call these Lie algebras *Poisson-determining*, with a modifier added to indicate a particular class of Poisson structures and coordinate transformations.

The first result about linearization is due to Arnol'd (see Appendix 14 of [6]), who showed that the 2-dimensional nonabelian Lie algebra is Poisson-determining in any category. Next, it was proved in [127] that any semisimple Lie algebra is formally Poisson-determining; i.e. a Poisson structure whose coefficients are formal power series can be linearized by a formal change of coordinates if the tangent Lie algebra is semisimple. An example in the same paper showed that the semisimple Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is not C^∞ Poisson-determining. In [130], it was shown that no semisimple algebra of noncompact type with real rank greater than 1 can be C^∞ Poisson-determining.

Some examples of nonlinearizable transverse structures to symplectic leaves in Lie–Poisson spaces are due to Givental' (see the discussion in [128]) and Damianou [26]. (Marí Beffa [10] and Ovsienko–Khesin [104], by contrast, show linearizability of transverse structure in some infinite-dimensional cases related to integrable systems.) Cahen–Gutt–Rawnsley [17] found nonlinearizability at the identity element for many Poisson Lie groups. (See Section 8.3.)

Positive results on Poisson-determinacy of semisimple algebras are due to Conn. He showed first [21] that any semisimple algebra is analytically Poisson-determining, and then [22] that any semisimple algebra of compact type is C^∞ Poisson-determining. These results were extended by Molinier [100] to the sum of semisimple algebras with $\mathbb{R}$.

There is a striking similarity between these results and those for the linearizability of actions of Lie algebras near a fixed point, so that it is tempting to try to derive the former results using the latter, whose proofs can be carried out quite simply by an averaging trick in the compact case, together with analytic continuation to reduce the noncompact analytic case to the compact one.

A global linearization theorem for the duals of compact semisimple Poisson Lie groups (see Section 8.3) was obtained by Ginzburg and the author in [49]. Another proof of this result was found by Alekseev [1] as part of a general theory which reduces actions of Poisson Lie groups to ordinary symmetry actions.

When the tangent Lie algebra $\mathfrak{g}$ is arbitrary, Wade [125] has shown that one can find a formal coordinate system which linearizes all the components of the Poisson tensor involving generators of the semisimple part of the Levi decomposition of $\mathfrak{g}$. Furthermore, Flato and Sternheimer (private communication) have pointed out that the linearization theorem in the semisimple case itself represents a Levi decomposition for the infinite dimensional Poisson bracket Lie algebra of germs, or of formal power series, since the functions vanishing at least quadratically at a totally degenerate point form a "topologically nilpotent" ideal with the tangent Lie algebra as quotient. So far, though, no proof of an analytic or C^∞ Poisson linearization theorem using this idea has been found.

There are by now many examples of Poisson-determining Lie algebras which are not semisimple. See, for example [32].

3.3 Quadratic Poisson structures and their perturbations

After the linear Poisson structures, it is natural to look at quadratic structures. It is perhaps surprising that these structures also arise "in nature," when Poisson structures on matrix Lie groups are extended to the matrix algebras which contain them [8, 111]. Two basic questions arise–classification and quadratization. A study of the classification was begun by Dufour–

Haraki [34] and Liu–Xu [85], and others. Quadratization (i.e. equivalence to quadratic structures after a coordinate change) has been established in some situations for structures with sufficiently nice quadratic part by Dufour [33] and Haraki [56].

4 Global structure of regular Poisson manifolds

The classification of regular Poisson structures on a given manifold M can be subdivided as follows.

1. Classify the foliations on M.
2. Classify the Poisson structures having a given foliation $\mathcal{F}$ as its symplectic leaf foliation.
    (a) Which foliations arise from some Poisson structure?
    (b) How many "different" Poisson structures can have the same foliation by symplectic leaves?

Problem 1 is obviously beyond the scope of this paper, so we will begin with a particular $\mathcal{F}$ and try to classify all the Poisson structures from which it can arise. In its full generality, Problem 2 includes the classification of symplectic manifolds, since M can always be foliated by a single leaf. This classification is complete only for 2-dimensional manifolds ([102] for the compact case, [52] for the noncompact case), although our knowledge in dimension 4 is rapidly growing through the combination of flexible constructions [50] and the rigidity results arising from Seiberg-Witten theory [117] (also see the extensive review of this paper [110]).

For foliations of dimension 2 on compact manifolds, Poisson structures which induce a given orientation on the leaves certainly exist, and they are classified by their classes in the second de Rham cohomology along the leaves [58]. One is thus led to the computation of this cohomology and the determination of the convex cone realized by Poisson structures. Some examples are worked out in [58]. The higher dimensional problem is virtually untouched.

5 Poisson cohomology and homology

Lichnerowicz [80] observed that the operation $[\pi, \ ]$ of Schouten bracket with a Poisson tensor is a differential on multivector fields, and he began the study of the resulting cohomology theory for Poisson manifolds. In particular, he showed that the map from differential forms to multivector fields determined by $\tilde{\pi} : T^*M \to TM$ is a morphism from the de Rham complex to the Poisson complex. In the symplectic case, this map is an isomorphism, but the Poisson cohomology spaces $H^i_\pi(M)$ are in general quite different from the de Rham cohomology.

As with most cohomology theories, the $H^i_\pi(M)$ have interesting interpretations for the first few values of i.

The differential in degree zero assigns to each function its hamiltonian vector field, so $H^0_\pi(M)$ consists of the functions which Poisson commute with everything, the so-called *Casimir functions* on M. It is suggestive to think of them as the "smooth functions on the space of symplectic leaves." The next differential maps each vector field X to $-\mathcal{L}_X\pi$, so $H^1_\pi(M)$ is the space of infinitesimal Poisson automorphisms modulo hamiltonian vector fields, or, algebraically, the outer derivations of the Poisson algebra. Ginzburg and Lu [48] make a case for thinking of $H^1_\pi(M)$ as the "space of vector fields on the space of symplectic leaves." Next, we can interpret $H^2_\pi(M)$ as the space of infinitesimal deformations of the Poisson structure modulo trivial deformations, while $H^3_\pi(M)$ receives the obstructions to extending infinitesimal deformations to formal deformations of higher and higher order.

These cohomology spaces have also been known for some time to relate to the deformation quantization of Poisson manifolds. For instance, $H^2_\pi(M)$ is, at least in the symplectic case, a parameter space for classifying deformation quantizations, while $H^3_\pi(M)$ receives the possible obstructions to constructing such quantizations. Kontsevich [69] shows that these relations on the cohomology level are simply the shadow of a much deeper relation between Poisson geometry and deformations of $C^\infty(M)$: the algebra of multivector fields, with zero differential, is quasi-isomorphic to the differential graded Lie algebra of multidifferential operators, with its Gerstenhaber [46] bracket, which controls the algebraic deformations.

Computation of Poisson cohomology is generally quite difficult. For regular Poisson manifolds, this cohomology reflects the topology of the leaf space and the variation in the symplectic structure as one passes from one leaf to another. Some references on the computation of this cohomology are

[47, 48, 120, 123, 138].

There is also a homology theory for Poisson manifolds, for which the chains are differential forms. The boundary operator δ was defined differential geometrically by Koszul [76] as $i_\pi d - di_\pi$, where i_π is the operator of contraction with the Poisson tensor. Brylinski [16] found an algebraic definition of the boundary operator by taking the classical limit of the Hochschild boundary operator for a quantized Poisson algebra. Brylinski also observed that δ becomes the "Hodge dual" of d when the manifold is symplectic, if one imitates Hodge theory by using the symplectic structure in place of a riemannian metric. Even when the manifold is not symplectic, the notion of a Poisson-harmonic form turns out to be remarkably useful; see the end of Section 8.5.

Huebschmann [59] studied both Poisson homology and cohomology from the algebraic point of view and showed how they could be fit in the standard homological framework of resolutions.

Some interesting identities relating the Poisson homology and cohomology differentials were found by Bhaskara and Viswanath [13] (see also [120]). More recently, the duality between the two theories has been studied in depth by Evens–Lu–Weinstein [39], Huebschmann [61], and Xu [141]. This work uses the fact that Poisson cohomology is a special case of Lie algebroid cohomology (see Section 7.1), while Poisson homology can be identified with Lie algebroid cohomology with coefficients in the top exterior power of the cotangent bundle. This identification establishes close relations between the homological theory and the modular theory described in Section 9. It is also related (see [61]) to the use of "dualizing sheaves" in algebraic geometry [57].

6 Completeness

There are several interesting notions of completeness in Poisson geometry. Their quantum analogues should be related to notions of self-adjointness.

6.1 Complete functions

A function $f : M \to \mathbb{R}$ on a Poisson manifold M will be called complete if the hamiltonian vector field X_f is a complete vector field, i.e. if its flow is globally defined.

Any compactly supported function is complete, and a riemannian metric on a manifold Q is complete iff the corresponding kinetic energy function on T^*Q is a complete function.

6.2 Complete manifolds

A Poisson manifold M will be called complete if every hamiltonian vector field on M is complete.

M is complete if and only if every symplectic leaf is bounded in the sense that its closure is compact. In fact, if every symplectic leaf is bounded, every trajectory of every hamiltonian vector field is contained in a compact set, so it can be continued for all time. Conversely, in any unbounded leaf $\mathcal{O}$ we can find a sequence of points which leaves every compact subset of M. By connecting these points with embedded paths, rounding corners, and removing intersections as necessary (a process which is easy when the dimension of $\mathcal{O}$ is more than 2 and harder but possible in the 2-dimensional case), we can find a proper embedding $\sigma : [0, 1) \to M$ whose image is contained in $\mathcal{O}$. Now choose a 1-form α along the image of σ so that $-\tilde{\pi}(\alpha(\sigma(t))) = \sigma'(t)$, and construct a function h on M whose differential at each $\sigma(t)$ is $\alpha(\sigma(t))$. The trajectory of the hamiltonian vector field X_h which begins at $\sigma(0)$ then follows σ but "reaches infinity" in a finite time, so X_h is not complete, and hence M is not complete.

A Lie–Poisson space $\mathfrak{g}^*$ is complete if and only if $\mathfrak{g}$ is the Lie algebra of a compact group. The sufficiency of the compactness condition is evident. For the necessity, we note that any group of transformations of a finite dimensional vector space with all orbits bounded is contained in a set of operators which is bounded, hence compact, so it admits an invariant, positive definite, inner product. (I would like to thank Marc Rieffel for providing me with this argument.) If $\mathfrak{g}^*$ is complete, all the coadjoint orbits are bounded, so the coadjoint representation admits an invariant, positive definite, inner product, and hence so does the adjoint representation. It follows that $\mathfrak{g}$ is the Lie algebra of a compact group. (See for example Section 21 of [99].)

6.3 Complete maps

A Poisson map $\phi : M \to N$ will be called complete if the pullback by ϕ of every complete function is complete.

It is a nice exercise to show that a complete map to $\mathbb{R}$ (with the zero Poisson structure, of course) is a complete function, and that to check completeness it suffices to look at the pullbacks of compactly supported functions. Also note that a Poisson manifold M is complete if and only if every Poisson map with domain M is complete. The inclusion of each symplectic leaf into any Poisson manifold is a complete map.

The composition of complete Poisson maps is complete, and the Poisson category becomes "tamer" when we restrict to complete maps. For instance, the image of a complete Poisson map $M \to N$ is always a union of symplectic leaves in N, so we do not have to deal with inclusions of arbitrary open subsets. In this sense, complete maps between Poisson manifolds are somewhat like proper maps between locally compact topological spaces. (Note that any proper Poisson map is complete.)

For the next two examples, we refer to [18] for further details.

If $N = \mathfrak{g}^*$ is a Lie–Poisson space, then each Poisson map $M \to N$ is the momentum map for a hamiltonian action of the Lie algebra $\mathfrak{g}$ on M. This action integrates to a hamiltonian action of the corresponding (connected) simply-connected Lie group G if and only if the momentum map is complete. A corollary of this result is that linear Poisson maps (which are just the duals of Lie algebra homomorphisms) are always complete.

There is an analogous result for complete Poisson maps $\phi : M \to N$ when N is a symplectic manifold. Any Poisson map from M to a symplectic manifold is a submersion, and the hamiltonian vector fields on M generated by functions pulled back from N span an integrable distribution on M. When the map ϕ is complete, integration of these vector fields for compactly supported functions on N can be used to construct local trivializations, which shows that $\phi : M \to N$ is a fibre bundle with a flat Ehresmann connection. The holonomy of this connection is an action of the fundamental group of N on the typical fibre of ϕ. Thus, there is a correspondence between complete Poisson maps to symplectic N and actions of the fundamental group $\pi_1(N)$, rather analogous to that between complete Poisson maps to $\mathfrak{g}^*$ and actions of G. It is tempting to think of the symplectic manifold N as the "dual of the Lie algebra of $\pi_1(N)$."

The analogy above is not quite precise because one needs to choose a basepoint in N to define the fundamental group and to specify the fibre on which it acts. It is actually better to think of the flat connection on $\phi : M \to N$ as corresponding to an action of the fundamental group*oid* of N on M. (See [55] for a treatment of flat connections on vector bundles as linear representations of fundamental groupoids.) The cases where N is Lie–Poisson or symplectic are then both subsumed under the theory of moment maps for actions of symplectic groupoids [97]. In fact, the correct interpretation of arbitrary complete maps $M \to N$ as momentum maps requires the use of such groupoids, which we discuss in Section 8.1.

6.4 Completeness of Poisson Lie groups

There is an independent notion of completeness for Poisson Lie groups. We introduce these groups in Section 8.3, along with the notion of dressing transformation. A Poisson Lie group G is said to be complete if the infinitesimal dressing transformations on the dual group G^* corresponding to the elements of the Lie algebra $\mathfrak{g}$ are complete vector fields.

A theorem of Majid [95] establishes that G is complete if and only if G^* is complete; it would be nice to have a Poisson-geometric proof of this result.

One can also ask when G is complete just as a Poisson manifold. The case $G = \mathfrak{g}^*$ shows that this is rather different from completeness as a Poisson Lie group; a Lie–Poisson space is always complete as a Poisson Lie group, since the dressing action on its dual is trivial, but as we have seen earlier in this section, such a space may or not be complete as a manifold.

There is more to be said about completeness and Poisson Lie groups, but we defer this to Section 8.4, after we have written more about these groups, and about symplectic groupoids.

6.5 Completeness in Poisson cohomology classes

Motivated by the example of the modular class (see Section 9 below), it is interesting to look at the completeness of Poisson vector fields which are not hamiltonian, looking at Poisson cohomology classes in $H^1_\pi(M)$ one at a time.

The zero class, i.e. the hamiltonian vector fields, always contains *some* complete vector fields, e.g. the zero field, but this may not be true for nonzero classes. For instance, on the "truncated" cylinder $(-1, 1) \times \mathbb{T}$ with symplectic structure $dy \wedge d\theta$, every complete, locally hamiltonian vector field is globally hamiltonian (Lemma 6.1 in [133]), so the nonzero classes in $H^1_\pi(M) = \mathbb{R}$ do not contain any complete vector fields at all. On the other hand, on the full cylinder $\mathbb{R} \times \mathbb{T}$, one can find both complete and incomplete vector fields in each cohomology class.

On the product $\mathbb{T}^2 \times \mathbb{R}$ with coordinates (θ_1, θ_2, t), we may compare the

Poisson structures

$$\frac{\partial}{\partial\theta_1}\wedge\frac{\partial}{\partial\theta_2} \text{ and } (1+t^2)\frac{\partial}{\partial\theta_1}\wedge\frac{\partial}{\partial\theta_2}.$$

In each case, there is a natural map τ from H^1_π onto the vector fields on the t-axis, and completeness of a Poisson vector field depends only on the image under τ of its cohomology class. For the first structure, τ is surjective since the leaves are all isomorphic as symplectic manifolds; as a result, there are many cohomology classes which contain no complete vector fields. For the second structure, τ is zero because the symplectic volume varies from leaf to leaf, so every Poisson vector field is complete.

It would be interesting to find a manifold on which Poisson vector fields are forced to be complete for reasons less trivial than in the previous example. Perhaps some "dynamical" property of the leaves could prevent trajectories from escaping to infinity in a finite time.

7 Lie algebroids and Lie groupoids

Up to now, when we have thought of a Poisson algebra $\mathcal{A}$ as an object with two structures, the multiplicative structure has been the primary one, and the Lie algebra structure secondary. But it is possible to take the opposite point of view, thinking of $\mathcal{A}$ first of all as a Lie algebra. This Lie algebra can in many cases be integrated to a Lie group $\mathcal{G}$ (of infinite dimension), and we may then ask what the associative algebra structure on $\mathcal{A}$ implies for $\mathcal{G}$. We will limit our attention to the case of Poisson manifolds, referring to [59] for general Poisson algebras.

As background references for the material on Lie algebroids and Lie groupoids in this section, we suggest Mackenzie's book [92] and the notes [18].

7.1 Lie algebroids

Instead of working directly with the Lie algebra $\mathcal{A} = C^\infty(M)$, we introduce a Lie algebra structure over $\mathbb{R}$ on the space $\mathcal{E}$ of 1-forms on a Poisson manifold M by the formula

$$[a,b] = \mathcal{L}_{\tilde{\pi}a}b - \mathcal{L}_{\tilde{\pi}b}a - d\pi(a,b),$$

where $\tilde{\pi}$ is the bundle map defined in Section 2. This bracket of 1-forms has the property $[df, dg] = d\{f, g\}$, so the Lie algebra $\mathcal{A}/\mathbb{R}$ appears as a subalgebra of $\mathcal{E}$.

The bracket on $\mathcal{E}$ is related to multiplication by functions through the Leibniz-type identity

$$[a, fb] = f[a, b] + (\rho(a) \cdot f)b, \tag{1}$$

where $\rho = \tilde{\pi}$.

For any commutative associative algebra $\mathcal{A}$ over $\mathbb{R}$, an $(\mathbb{R}, \mathcal{A})$ *Lie algebra* is a Lie algebra $\mathcal{E}$ over $\mathbb{R}$ carrying the additional structure of an $\mathcal{A}$-module, together with a map ρ from $\mathcal{E}$ to the derivations of $\mathcal{A}$ which is a homomorphism for both the $\mathcal{A}$-module and Lie algebra structures, and which satisfies identity (1) above. When $\mathcal{A} = C^\infty(M)$ and $\mathcal{E}$ is the $C^\infty(M)$-module of sections of a vector bundle E, the map ρ is realized by a bundle map from E to TM, which we will also denote by ρ. The bracket on sections of E together with the map ρ, called the *anchor*, is called a *Lie algebroid* structure on E.

Thus the cotangent bundle T^*M of a Poisson manifold M is a Lie algebroid in a natural way. The bracket on 1-forms on M was discovered independently by many people, beginning apparently with Fuchssteiner [43]. The Lie algebroid structure on T^*M was first used in [24, 129].

There is also a connection in the reverse direction between Poisson manifolds and Lie algebroids: the dual vector bundle E^* of any Lie algebroid carries a natural Poisson structure [24, 25]. Like the Lie–Poisson structure on the dual of a Lie algebra, it is determined by the brackets among a small class of functions–in this case, the fibrewise affine functions on E^*, which are identified with sections of E plus functions on M.

Besides the cotangent bundles of Poisson manifolds, there are many other interesting examples of Lie algebroids. The tangent bundle TM of any manifold is a basic example, isomorphic as a Lie algebroid to T^*M when M is a nondegenerate (i.e. symplectic) Poisson manifold. Other examples are integrable subbundles of TM, Lie algebras $\mathfrak{g}$ (which are Lie algebroids over a point), and product bundles $M \times \mathfrak{g}$ for manifolds M carrying actions of $\mathfrak{g}$, with a bracket on $C^\infty(M, \mathfrak{g})$ induced from the action.

It can be useful to think of a Lie algebroid E over M as a "new tangent bundle" for M. The algebra $\Omega_E(M)$ of multilinear alternating forms on E then plays the role of differential forms for M with this new structure. In fact, the Lie algebroid properties enable one to define a differential d_E on the algebra $\Omega_E(M)$ which makes it into a complex whose cohomology is known as *Lie algebroid cohomology.* Special cases of this cohomology include de Rham cohomology ($E = TM$), Poisson cohomology ($E = T^*M$), Chevalley cohomology of Lie algebras (M is a point), and leafwise de Rham cohomology of foliations (E is the tangent bundle along the leaves). In fact, the Lie algebroid structure on E is essentially equivalent to the differential d_E on $\Omega_E(M)$. In this way, a Lie algebroid can be viewed as a supermanifold (the space on which $\Omega_E(M)$ is the functions) carrying an odd vector field with square zero (the derivation d_E) [2, 119].

7.2 Lie groupoids

Since an $(\mathbb{R}, \mathcal{A})$ Lie algebra $\mathcal{E}$ acts by derivations on the algebra $\mathcal{A}$, a Lie group $\mathcal{G}$ whose Lie algebra is $\mathcal{E}$ should act on $\mathcal{A}$ by automorphisms, at least when some completeness condition is satisfied. In addition, the $\mathcal{A}$ module structure on $\mathcal{E}$ should be reflected in some further structure on $\mathcal{G}$.

We shall limit ourselves here to the geometric situation where $\mathcal{E}$ is the space of sections of a Lie algebroid E over a manifold M. The $C^\infty(M)$ module structure on E is "integrated" to the fact that $\mathcal{G}$ is itself a space of sections of a "bundle" $\beta : G \to M$, and the action of $\mathcal{G}$ on $C^\infty(M)$ becomes an action by diffeomorphisms of M, which is encoded geometrically by a second map $\alpha : G \to M$. The structure on the manifold G is that of a *Lie groupoid.*

To be precise, we recall first that a *small category* G over a *base* M is a set G equipped with *source* and *target* maps β and α from G to M, a *unit section* $\epsilon : M \to G$, and a *multiplication* operation $(x, y) \mapsto xy$ defined on the set $G * G = \{(x, y) \in G \times G | \beta(x) = \alpha(y)\}$ of *composable pairs.* These operations satisfy the conditions that $\alpha(xy) = \alpha(x)$, $\beta(xy) = \beta(y)$, $(xy)z = x(yz)$ when either side is defined, and the elements of $\epsilon(M)$ in G act as identities for multiplication. If all the elements of G have inverses with respect to these identities, G is called a *groupoid.* If G and M are manifolds, and the structural maps are smooth (one requires α and β to be submersions to insure that the domain of multiplication is a manifold), then G is called a *Lie groupoid.*

The Lie algebra of vector fields on a Lie groupoid G contains a distinguished subalgebra $\mathcal{E}$ of fields which are left-invariant in a certain sense; these are the sections of a vector bundle E which can be identified with the normal bundle to $\epsilon(M)$ in G, and then with the kernel of $T\alpha$. $T\beta$ is then an anchor map $E \to TM$ for a Lie algebroid on E. We call this *the Lie* *algebroid of the Lie groupoid G.*

As one knows already from the case of groups (groupoids for which M has one element), there may be nonisomorphic groupoids having isomorphic Lie algebroids–the groupoids are distinguished by the zeroth and first homotopy groups of their α-fibres. But unlike (finite dimensional) Lie algebras, not all Lie algebroids can be integrated to Lie groupoids [3]. They can be integrated [107] to *local Lie groupoids*, the natural candidates for a neighborhood of $\epsilon(M)$ in a Lie groupoid.

In any [local] Lie groupoid G over M, the set $S(G)$ of submanifolds which project diffeomorphically onto M under both α and β (they are called *bi-sections*) inherits a [local] group structure from the multiplication on G. The Lie algebra of this group (defined in terms of its 1-parameter subgroups) can be identified with the sections of the Lie algebroid of G.

8 Poisson groupoids and their actions

8.1 Symplectic groupoids

We have seen that the cotangent bundle T^*M of a Poisson manifold has a natural Lie algebroid structure derived from the Poisson bracket of functions. If there is a Lie groupoid G whose Lie algebroid is isomorphic to T^*M, we say that M is an *integrable* Poisson manifold.

When the fibres of α are connected and simply connected (and sometimes even when they are not), the canonical symplectic structure on T^*M induces on G a symplectic structure for which the multiplication is symplectic in the sense that its graph $\{(z, x, y)|z = xy\}$ is a lagrangian submanifold of the product $G \times \overline{G} \times \overline{G}$. This property of multiplication implies that the set $LS(G)$ consisting of the lagrangian bi-sections is a subgroup of $S(G)$, as is its subgroup $HLS(G)$ consisting of the lagrangian bi-sections obtained from $\epsilon(M)$ by hamiltonian deformations. The Lie algebras of these two groups consist of the closed and exact 1-forms respectively. In particular, $HLS(G)$ is a group whose Lie algebra is $C^\infty(M)/\mathbb{R}$. There is a natural action of $HLS(G)$ on M by Poisson automorphisms which can be considered as "inner," since they are generated by hamiltonian vector fields. We refer to [140] for further discussion of these groups.

The target map $\alpha : G \to M$ of a [local] symplectic groupoid is always a Poisson map. This shows that any Poisson manifold can be realized as the quotient of a symplectic manifold by a foliation compatible with the symplectic Poisson brackets. Such "symplectic realizations" were first studied by Lie [82], who used them to prove his "third theorem": the existence of a local Lie group corresponding to any finite dimensional Lie algebra. Global realizations for arbitrary Poisson manifolds were first found by Karasev [64] and the author [129].

8.2 Groupoid actions

A groupoid G over a set M can act on a space Q equipped with a map $J : Q \to M$, which we call the *moment map* of the action. By definition, the action is a map $(x, q) \mapsto xq$ from $G * Q = \{(x, q) | \beta(x) = J(q)\}$ to Q satisfying the condition $\alpha(xq) = \alpha(x)$ and the usual laws for an action. For instance, G acts on itself by left multiplication with $J = \alpha$ and on M with J the identity. A groupoid action of G on Q induces a group action of $S(G)$ on Q.

If G is a symplectic groupoid for the Poisson manifold M, a G action on a Poisson manifold Q is a *Poisson action* if its graph $\{(r, x, q) | r = xq\}$ is a coisotropic submanifold of $Q \times \overline{G} \times \overline{Q}$. The moment map $J : Q \to M$ is then a Poisson map. This statement is also true for "local actions." Conversely, we have the following theorem of Dazord [27] and Xu [137].

Theorem 8.1 *Let G be a local symplectic groupoid for M. Every Poisson map $J : Q \to M$ is the moment map for a unique local action of G on Q. If M is integrable and G is its α-connected and α-simply connected (global) symplectic groupoid, then J is the moment map for an action of G on Q if and only if it is complete. The action is unique.*

This theorem unifies the two examples of complete Poisson maps which were classified in Section 6. When M is a Lie-Poisson space $\mathfrak{g}^*$, G is T^*G_0, where G_0 is the connected and simply connected Lie group whose Lie algebra is $\mathfrak{g}$. Poisson actions of G then correspond to hamiltonian actions of G_0 together with their momentum maps. When M is symplectic, G is $\pi(M)$, the *fundamental groupoid* of M consisting of homotopy classes of paths with fixed endpoints. An action of G is just a flat connection on a fibre bundle over M; it is a Poisson action when the fibre is a Poisson manifold and the parallel translations are Poisson maps.

To close this section, we note that Zakrzewski [142] has introduced a notion of morphism between symplectic groupoids such that each morphism induces a complete Poisson map between the underlying Poisson manifolds.

The morphisms are not in general functors or even mappings between the groupoids considered as categories, but are rather Poisson relations which are lagrangian submanifolds compatible with the groupoid structures.

8.3 Poisson Lie groups and groupoids

Symplectic groupoids appeared in the 1980's with the independent work of Karasev [64], Zakrzewski [142], and the author [129], motivated by quantization problems. Meanwhile, a theory of Poisson Lie groups had been developing through the work of Drinfel'd [29] and Semenov-Tian-Shansky [111, 112] on completely integrable systems and quantum groups. It was therefore natural to unify these two theories with a notion of Poisson groupoid [131]. A *Poisson groupoid* (for consistency, we should probably say "Poisson Lie groupoid") is a Lie groupoid G with a Poisson structure for which the graph of multiplication is a coisotropic submanifold of $G \times \overline{G} \times \overline{G}$. If G happens to be symplectic, then the graph of multiplication is forced to be lagrangian, and we have a symplectic groupoid as defined above.

The base M of a Poisson groupoid G has a unique Poisson structure making the target and source maps Poisson and anti-Poisson respectively, but this structure no longer determines the local groupoid as it did in the symplectic case. For example, if M is a point, G can be any Poisson Lie group.

The infinitesimal object which encodes the local structure of a Poisson groupoid is a *Lie bialgebroid*. This is a Lie algebroid E for which the dual vector bundle E^* also carries a Lie algebroid structure which is compatible in a certain way with that on E. The compatibility condition was first determined by Mackenzie and Xu [93] and then reinterpreted by Kosmann-Schwarzbach [71] in the language of Gerstenhaber algebras (a form of graded Poisson algebras). Mackenzie and Xu [94] proved that, if E is the Lie algebroid of a groupoid G with connected and simply connected α-fibres, then a compatible Lie algebroid structure on E^* always comes from a Poisson groupoid structure on G.

All of this structure was discovered much earlier [29] in the case of a Poisson Lie group G. Multiplication in such a group is a Poisson map from $G \times G$ to G. The Poisson tensor vanishes at the identity element, so that $\mathfrak{g}^*$ has the tangent Lie algebra structure. The corresponding Lie group, called the *dual Poisson Lie group* to G and denoted by G^*, is uniquely determined only locally; this ambiguity is frequently ignored in the literature.

The compatibility condition between Lie algebra structures on $\mathfrak{g}$ and $\mathfrak{g}^*$ in a *Lie bialgebra* is that the dual of the second structure is a Lie algebra 1-cocycle for $\mathfrak{g}$ with values in the second exterior power of the adjoint representation. This condition has another very useful formulation: the direct sum $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ carries a unique Lie algebra structure for which the natural symmetric bilinear form (in which the summands are isotropic) is invariant under the adjoint representation and the summands are subalgebras with the given structures. The algebra $\mathfrak{d}$ is called the *double* of the Lie bialgebra, and the corresponding (connected, simply connected) Lie group D is the double of the Poisson Lie group G. Any Lie algebra $\mathfrak{d}$ provided with an invariant symmetric bilinear form and a complementary pair of isotropic subalgebras is called a *Manin triple*; these objects are equivalent to Lie bialgebras.

It turns out [112] that an open dense subset of D (equal to all of D in some cases) carries a natural symplectic structure, and that there is a Poisson projection from this open set to G. This projection resembles the target map of a symplectic groupoid, and in fact Lu and Weinstein [90] have shown that a slight modification of D produces a symplectic manifold $\tilde{D}$ which carries a pair of "commuting" symplectic groupoid structures with bases G and G^*. This *symplectic double groupoid* is in a sense a geometric model for a Hopf algebra, or "quantum group," of which the bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ represents the classical limit. (Zakrzewski [142] replaces one of the the groupoid products by a "coproduct" to model more faithfully the Hopf algebra structure.) In [135], the double groupoid is used in a geometric analog of part of the construction of knot invariants from Hopf algebras; in particular, many "set theoretic solutions of the quantum Yang-Baxter equation" are found, answering a question of Drinfel'd [30]. The doubles of general Lie bialgebroids are discussed in Section 11.2.

8.4 Poisson group actions

If G is a Poisson groupoid over the Poisson manifold M, a G action on a Poisson manifold Q is a *Poisson action* if its graph $\{(r, x, q) | r = xq\}$ is a coisotropic submanifold of $Q \times \overline{G} \times \overline{Q}$. As in the symplectic case, the moment map $J : Q \to M$ is necessarily a Poisson map. We have already examined such actions when G is symplectic; now we will look at the case where G is a group.

An action on Q of a Poisson Lie group G is a Poisson action when the "action map" $G \times Q \to Q$ is a Poisson map. Each element x of G acts on

Q as a diffeomorphism, but this diffeomorphism is not necessarily a Poisson map.

Recall that, when G has the zero Poisson structure (so that a Poisson action is just an action by Poisson automorphisms), certain Poisson actions of G are distinguished as "hamiltonian." These are generated by momentum maps to $\mathfrak{g}^*$ and induce actions of the symplectic groupoid T^*G of $\mathfrak{g}^*$. This theory of momentum maps has been generalized by Lu [86] to the case of general Poisson Lie groups. Her momentum maps take values in the dual group G^* and exist for all Poisson actions on simply connected symplectic manifolds, and for many other actions as well, which we than call "hamiltonian" as before. One can lift these actions to the symplectic groupoid $\tilde{D}$ of G^*. Conversely, if G is connected and simply connected, by Theorem 8.1 above, one can integrate an arbitrary Poisson map $J : Q \to G^*$ to a local action of $\tilde{D}$ on Q, which is global if J is complete. Now, if G is complete as a Poisson group (see Section 6.4), $\tilde{D}$ contains a copy of G naturally embedded in its group of bi-sections, so there is an induced action of G on Q. These bi-sections are not lagrangian, so the action is not by Poisson maps; rather it is a Poisson action–in fact, it is the action whose momentum map is J.

The identity map on G^* is an important special case. The local action of G on G^* generated by this map is called the *dressing action*. (The term originated in the theory of completely integrable differential equations (see [112]), in which "dressing transformations" were used to build complicated solutions from simpler ones.) This action is global if and only if G^* is a complete Poisson Lie group in the sense of 6.4. This shows that a Poisson map $J : Q \to G^*$ is guaranteed to be the momentum map of a global Poisson action of G only if J is a complete map *and* G is a complete Poisson Lie group.

8.5 Poisson homogeneous spaces

Transitive Poisson actions of a Poisson Lie group G were identified by Drinfel'd [31] with maximal isotropic subalgebras of the double $\mathfrak{d}$ of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. This result has been applied by Karolinsky [66] to the classification of Poisson homogeneous spaces of compact semi-simple Lie groups with the standard Poisson Lie group structure. Lu [88] has obtained a similar classification and, in addition, has shown that these homogeneous spaces correspond to solutions of the so-called *classical dynamical Yang-Baxter equation* [41]. It is quite intriguing that these solutions have also been shown by Etingof–Varchenko [37] to produce examples of Poisson groupoids. while Lu [87] has found a more conceptual approach to Drinfeld's identification by connecting Poisson homogeneous spaces directly to Lie algebroids. A very beautiful theory should appear when the connections among all these results are sorted out.

The homogeneous spaces of compact semisimple groups known as *flag manifolds* carry very interesting homogeneous Poisson structures for which the symplectic leaves are the cells of the Bruhat (or Schubert) decomposition [91]. These Poisson structures have been analyzed in great detail by Evens and Lu [38, 89], who use them to explain geometrically some hitherto mysterious calculations by Kostant [74] of cohomology in Lie algebras and flag manifolds. In particular, they show that Kostant's "harmonic forms" are precisely the forms which are Poisson-harmonic in the sense of Section 5. Their work suggests the possibility of using Poisson structures as a tool for studying the topology of singular spaces. (I owe this remark to Sam Evens and J.-H. Lu.)

8.6 Moment maps vs. momentum maps

The terms "moment map" and "momentum map" are usually used interchangeably in the literature, with different authors preferring each of these two translations of Souriau's [116] French term, "moment." By contrast, in this paper, we have used the terms in different ways. For us, a "momentum map" is a Poisson map $J : Q \to G^*$ to the dual of a Poisson Lie group G, generating a hamiltonian action of G on Q. On the other hand, a "moment map" is a map to the base of a groupoid which is acting on Q; for a Poisson groupoid action, the moment map is a Poisson map.

These two objects are related in the following way. The momentum map $J : Q \to G^*$ for a hamiltonian action of a Poisson group G is also the moment map for the corresponding Poisson groupoid action of the symplectic groupoid action of $\tilde{D}$ on Q. But some pieces of the puzzle are missing here. Among the Poisson actions of a Poisson groupoid G on Q, there should be some which are distinguished as "hamiltonian," having momentum maps to the dual Poisson groupoid G^* (defined as in the case of groups by integrating the dual of the Lie algebroid, when this is possible). These momentum maps should then generate actions on Q by a symplectic groupoid $\tilde{D}$ of G^*. A first look at the problem suggests that hamiltonian actions may be rare for groupoids which are not groups.

9 Modular theory

The *modular automorphism group* of a von Neumann algebra $\mathcal{A}$ is a one-parameter group of automorphisms of $\mathcal{A}$ which is generated by a weight of $\mathcal{A}$. It is the trivial group whenever the weight is a trace. This group was first constructed by Tomita, who was motivated by the special case of the convolution algebra of a locally compact group, on which the modular function measures the discrepancy between left- and right-invariant measures. An exposition by Takesaki rendered Tomita's work much more accessible, and the modular theory is now known as *Tomita–Takesaki theory.* We refer to Section V.3 of Connes' book [23] for further references to this theory and its connections with the KMS (Kubo–Martin–Schwinger) equilibrium conditions in statistical mechanics.

The modular theory was a basic ingredient in Connes' classification theory for type III von Neumann algebras. In particular, Connes sharpened the theory by proving a "Radon–Nikodym theorem" to the effect that the modular automorphism groups generated by two different weights are equal modulo inner automorphisms. As a result, each von Neumann algebra $\mathcal{A}$ carries a well-defined one-parameter subgroup of its outer automorphism group. This group is trivial if and only if $\mathcal{A}$ admits a trace.

When the algebra $\mathcal{A}$ is commutative, every weight is a trace, and so the modular automorphism group is always trivial. Nevertheless, Gallavotti and Pulvirenti [44] observed that the modular ideas could be applied to classical statistical mechanics if a subalgebra $\mathcal{A}$ of the complex-valued continuous functions $C(M)$ on a topological space M is equipped with a skew-symmetric bracket, with values in $C(M)$, satisfying some axioms resembling those of a Poisson bracket (not including the Jacobi identity!). They define a *state* in this context to be a probability measure μ on M such that $\mathcal{A}$ is contained and dense in $L^2(M, \mu)$. The measure μ is the classical analog of a weight, and the analog of the trace condition is that it should vanish when applied (via integration) to any Poisson bracket $\{f, g\}$.

Gallavotti and Pulvirenti then impose some further conditions so that the skew-hermitian bilinear form $\mu\{\overline{f}, g\}$ on $\mathcal{A}$ is defined and is realized by a densely defined operator X_μ for which iX_μ is essentially self-adjoint. X_μ is a derivation (generally unbounded); the main theorem in [44] states that this derivation generates a one-parameter group of automorphisms of $L^\infty(M, \mu)$ coming from a one-parameter group of automorphisms of the measure space (M, μ).

The analysis just described becomes very simple when M is a Poisson manifold and $\mathcal{A}$ is the algebra of smooth functions. X_μ is a smooth Poisson vector field when μ is a smooth measure. The construction of this *modular vector field* is local, so there is no reason to require μ to be of total measure 1, or even finite.

This vector field X_μ was constructed by Koszul [76] for reasons totally unconnected with operator algebras or statistical mechanics (though he did observe that, for M a Lie–Poisson space $\mathfrak{g}^*$ and μ a translation-invariant measure, X_μ is the infinitesimal modular character defined as the trace of the adjoint representation). It was used as a tool for the classification of Poisson structures by Dufour–Haraki [34] and others [51, 85].

Stimulated by the modular theory in operator algebras, Weinstein [133] rediscovered the vector field X_μ by defining it as the operator which assigns to each function f the divergence with respect to μ of its hamiltonian vector field X_f. He observed that a change in μ simply adds to X_μ a hamiltonian vector field, the analogue in Poisson terms of an inner derivation. Thus, the class of X_μ modulo hamiltonian vector vector fields is a well-defined element of the Poisson cohomology space $H^1_\pi(M)$ which may be called the *modular class* of M.

The existence of the modular automorphism *group* of a Poisson manifold M just depends on whether some representative of the modular class is a complete vector field. Some examples where this may or may not be the case are given in [133].

When M is a symplectic manifold, the modular vector field is zero when μ is the symplectic measure. Thus, all modular vector fields are hamiltonian, and when the correspondence "measure $\to$ hamiltonian function" is reversed, one obtains the Gibbs measure associated to each hamiltonian system. This aspect of classical KMS theory was investigated by Basart, Flato, Lichnerowicz, and Sternheimer [9] in the setting of conformal symplectic geometry and deformation theory.

The modular class for Poisson manifolds is a special case of a construction for Lie algebroids and groupoids [39] and even for $(\mathbb{R}, \mathcal{A})$ Lie algebras [61], where it plays a key role in Poincaré duality theory.

10 Self-similarities and Liouville vector fields

A *dilation* of a Poisson manifold M is a pair (ϕ, λ), where λ is a nonzero real number and $\phi : M \to M$ is a diffeomorphism such that $\{\phi^* f, \phi^* g\} = \lambda\phi^*\{f, g\}$ for all f and g in $C^\infty(M)$. Equivalently, $\phi^*\pi = \lambda^{-1}\pi$. The dilations of M form a group, and the image $F(M)$ of this group under the projection to the multiplicative group $\mathbb{R}^\times$ of nonzero real numbers will be called the *Murray–von Neumann fundamental group* of M, by analogy with a similar invariant for von Neumann algebras (it is the group denoted by $F(N)$ on p. 457 of [23]), which is sometimes referred to in that subject simply as the "fundamental group" of a von Neumann algebra. This similarity was called to our attention by D. Shlyakhtenko [114].

$F(M)$ contains all the positive real numbers when M admits a complete *Liouville vector field*, i.e. a vector field X for which $[X, \pi] = -\pi$. This definition extends the one where π comes from a symplectic form ω, in which case X is required to satisfy $\mathcal{L}_X\omega = \omega$. The existence of such X in the symplectic case is equivalent to the exactness of the form ω, and in the general Poisson case to the exactness of π as a cocycle for Poisson cohomology. We therefore call a Poisson manifold admitting a Liouville vector field *exact*.

Although a compact symplectic manifold cannot be exact, examples in [58, 136] show that compact Poisson manifolds can be exact. The meaning of this condition in deformation quantization is discussed in [136], the last section of which also contains some examples of manifolds for which $F(M)$ is countable. For a translation-invariant Poisson structure on a torus, the fundamental group turns out to be closely related to the group of units in the integers of an algebraic number field [98].

There is a related notion for Poisson manifolds M which have vanishing modular class, and admitting only constant Casimir functions. Such a manifold admits a unique ray of smooth invariant measures. For every Poisson automorphism ϕ of M, there is a unique positive real number ν such that, for any smooth invariant measure μ on M, $\mu(\phi^* f) = \nu\mu(f)$ for all f in $C^\infty(M)$. The set of all such ν as ϕ ranges over the automorphism group is the *trace scaling group* of M.

Finally we mention another connection between the fundamental group and the modular automorphism group. If (ϕ, λ) is a dilation of M and μ is a smooth measure on M, then $\phi^* X_\mu = \lambda^{-1} X_{\phi^*\mu}$, so that ϕ^* multiplies the modular class by λ^{-1}. An infinitesimal version of this argument shows that, if X is a Liouville vector field, then $[X, X_\mu] + X_\mu$ is a hamiltonian vector field. These facts can be used to show the nonexistence of nontrivial dilations or Liouville vector fields under certain conditions, such as in the following example.

On the Poisson Lie group $SU(2)$ with the Bruhat–Poisson structure, the modular vector field for any measure is nonzero and tangent to the circle of diagonal matrices. The period of this vector field is an invariant under diffeomorphisms; hence -1 is the only possible nontrivial element of the fundamental group (it is realized by the group inversion map, as is the case on any Poisson–Lie group), and there are no Liouville vector fields. A similar argument applies to the reduced Poisson structure on the flag manifold S^2. This structure has the form $\{x, y\} = x^2 + y^2$ in local coordinates near one point; the linearized modular vector field there is a rotation with period 2π. J.-H. Lu (private communication) has shown that these arguments can be generalized to all the compact groups with Bruhat–Poisson structure. She proves that, if the modular class is non-zero, and one of its representatives acts semi-simply on the multivector fields on M, then a Liouville vector field cannot exist. (A lemma for this proof is that a modular vector field always acts trivially on Poisson cohomology.) She then uses an explicit description [39] of the modular vector fields of Bruhat–Poisson structures to show that they act semi-simply.

It is interesting to contrast the results above with the result of Sheu (private communication) that the Bruhat–Poisson Lie groups $SU(n)$, all admit *continuous* Poisson dilations with all possible scaling factors. (These maps are homeomorphisms which are symplectic dilations on all symplectic leaves.) It is not known whether such dilations exist for the Bruhat–Poisson structures on the other series of classical groups.

11 Generalizations

There are two ways in which the notion of Poisson manifold can be usefully generalized. The first is to keep the Poisson algebra idea, but to allow the underlying space to be something other than a differentiable manifold. The second is to alter the axioms for the Poisson bracket itself.

11.1 Poisson spaces

The derivation property of the operations $\{\ ,h\}$ on a Poisson algebra $\mathcal{A}$ suggests that, when $\mathcal{A}$ is an algebra of functions on a space, this space should have some kind of "differential structure." Even on a Poisson manifold, though, it is not really necessary to differentiate except along the symplectic leaves. This suggests that there may be useful notions of "Poisson topological space," or even "Poisson measurable space," specified by Poisson algebras which are dense in the continuous or measurable functions on a topological or measure space M. We have already mentioned (see Section 9) work in this direction by Gallavotti and Pulvirenti [44].

Poisson topological spaces seem to occur naturally as classical limits of C^*-algebras. For instance, Sheu [113] constructs an algebra of functions on a quantum $SU(2)$ by deforming a Poisson algebra of continuous functions on the classical $SU(2)$. Landsman [78] has introduced a notion of Poisson space designed to serve the purposes of *both* classical and quantum mechanics.

A guide to the measurable case, which should provide classical models for some von Neumann algebras, may be the notion of "foliated space" used by Moore and Schochet [101] in their study of index theory for foliations. For example, the union M_0 of the bounded symplectic leaves on a Poisson manifold (see Section 6.2), i.e. the "complete" part of M, is generally neither open nor closed, but it is measurable. More generally, there should be a decomposition of any Poisson manifold into "ergodic components" of its symplectic leaf foliation. These components will generally be measurable rather than smooth spaces.

There are also singular spaces arising from compact group actions. For example, the notion of "stratified symplectic space," developed by Lerman and Sjamaar [115] plays an essential role in the theory of reduction by compact group actions (see [68]). An important application of this theory is to moduli spaces of connections (see the paper of Huebschmann [60]). Egilsson [36] has looked at the quotient spaces of $\mathbb{R}^{2n}$ by symplectic circle actions and shown that, in certain cases, these singular Poisson varieties cannot be embedded into Poisson manifolds. A general study of Poisson algebraic geometry was begun by Berger [12], and Saint-Germain [109] has applied the algebraic point of view to Lie–Poisson geometry for nilpotent algebras, while Polishchuk [106] has considerably developed the theory in the complex case, where the usual rigidity associated with compact complex manifolds leads to very strong classification results. Finally, Farkas and Letzer (see fale:ring and several other recent papers by Farkas) study Poisson structures on commutative and noncommutative rings.

11.2 Other brackets

In the same paper [67] where he characterized Poisson structures, Kirillov also introduced more general brackets on sections of line bundles. These structures, dubbed *Jacobi structures* by Lichnerowicz [81], include as special cases Poisson structures, contact structures, conformal symplectic structures, and foliations with leaves of all these types. They naturally arise from Poisson structures as quotients by the flows of Liouville vector fields (see Section 10 above). Their local structure was analyzed in [28].

In the geometric study of completely integrable systems, an important role has been played by pairs of Poisson structures which are compatible in the sense that their sum is again a Poisson structure. The theory of so-called *bi-hamiltonian structures* has been extensively developed. We refer to the papers of Kosmann-Schwarzbach and Magri [73] and Vaisman [121] for discussion of the related *Poisson–Nijenhuis strucures* from a viewpoint close to that of this survey. This papers also include references to earlier work on bi-hamiltonian structures.

Although quotients of Poisson manifolds by groups of Poisson automorphisms (and even by Poisson group actions) inherit Poisson structures, passing to submanifolds takes us to a larger category, that of Dirac structures. Roughly speaking, a Dirac structure is a singular foliation together with a Poisson structure on the leaf space. More precisely, a Dirac structure is defined by a subbundle L of $TM \oplus T^*M$ which is maximal isotropic for the natural symmetric bilinear form, and which satisfies an integrability condition discovered by Courant [25]. When L is the graph of a map from T^*M to TM, the map is $-\tilde{\pi}$ for a Poisson structure π. When L is the graph of a map from TM to T^*M, the map comes from a closed 2-form. In general, the projection of L to T^*M defines a (singular) characteristic foliation, with a Poisson bracket defined on the leaf space. The projection of L to TM defines another singular foliation whose leaves carry closed (but possibly degenerate) 2-forms.

Courant's integrability condition is that the sections of L should be closed under a natural bracket which he constructed on the sections of $TM \oplus T^*M$. This suggests that the theory of Dirac structures should be a special case of a Lie algebroid version of Drinfel'd's theory of Poisson homogeneous spaces (see Section 8.5). However, although the pair (TM, T^*M), with the usual bracket on vector fields and the zero bracket on 1-forms, does form a Lie bialgebroid, Courant's bracket satisfies the Jacobi identity only modulo a "coboundary term." Thus, the double of a Lie bialgebroid is not a Lie algebroid, but rather a new kind of object called a *Courant algebroid*. The theory of these objects, resulting in a Lie algebroid version of Manin triples and Poisson homogeneous spaces (of Poisson groupoids) was developed by Liu, Weinstein, and Xu [83, 84]. In addition, Roytenberg and Weinstein [108] have interpreted Courant algebroid brackets as strongly homotopy Lie algebra structures [77], a general class of bracket operations on complexes, which satisfy the Jacobi identity modulo coboundary terms.

Poisson brackets which do not quite satisfy the Jacobi identity are also quite relevant to the study of mechanical systems with non-holonomic constraints. See, for example, the work of Koon and Marsden [70].

Another interesting variation on the notion of Poisson algebra is that of graded Poisson algebra, with various sign conditions for the commutativity of the product and anti-commutativity of the bracket. This leads both to Gerstenhaber algebras [62, 71, 141] and to other types of super-Poisson algebras and manifolds [19].

Finally, there is the idea of Nambu [103] of describing physical systems by an antisymmetric bracket on more than two variables in the algebra of observables. Both classical and quantum aspects of this *Nambu mechanics* have been pursued vigorously in recent years; see the survey by Flato, Dito, and Sternheimer [42].

12 Odds and ends

This section collects some miscellaneous examples and questions about Poisson geometry.

In answer to a question raised in an early version of this paper, Zakrzewski [144] used ideas from [143] to construct examples of Poisson structures on any $\mathbb{R}^{2n}$ having just two symplectic leaves: the origin and its complement. It turns out that a simple way to get some of Zakrzewski's examples is to begin with the standard symplectic structure on $\mathbb{R}^{2n}$ and then perform an inversion through the unit sphere. The resulting Poisson structure on the complement of the origin turns out to have coefficients which are quartic polynomials and which therefore extend smoothly over the whole space.

Using the inversion again, one can easily extend these structures to give Poisson structures on the even-dimensional spheres with just two symplectic leaves: a point and its complement. It should be interesting to study these examples further, finding their Poisson cohomology, quantizations, etc.

The examples above are special members of the class of Poisson manifolds for which the decomposition into symplectic leaves is locally finite. The Bruhat–Poisson structures on flag manifolds (see Section 8.5 above) are examples of such structures, and others can be constructed on toric varieties, by pushing forward a nondegenerate bivector field on the Lie algebra of a complex torus. Are these examples "typical" in any sense? Is there an interesting generic class of such Poisson structures? How are they related to the stratified symplectic manifolds of Sjamaar–Lerman [115]? Sam Evens and J.-H. Lu (private communication) have suggested that Poisson structures with locally finite symplectic leaf decomposition may be a useful geometric tool for studying the topology of the spaces on which they are defined, with [38] being only a model example.

There seems to be a close relation between the Atiyah–Guillemin–Sternberg [7, 53] theorem on the convexity properties of momentum maps and the Duistermaat–Heckman theorem [35] on the variation of the symplectic structure of reduced spaces with the value of the momentum map. (See Kirwan's survey [68] for generalizations to actions of nonabelian groups.) Both theorems refer to a hamiltonian action of a torus T on a symplectic manifold M. The first concerns the momentum map $M \to \mathfrak{t}^*$, while the second concerns the "dual" map $M \to M/T$. Since variation of the cohomology classes of the symplectic forms on leaves of a Poisson manifold is related to Poisson cohomology, it may be that ideas of Morita equivalence [48, 137] could be used to establish some duality between these two results.

In the paper [118], Terng proved a convexity theorem for projections of isoparametric submanifolds of euclidean spaces on their normal spaces. This theorem in riemannian geometry includes (and was suggested by) the convexity theorem for projections of coadjoint orbits of compact Lie groups onto to the dual of a Cartan subalgebra. Is there some Poisson geometry behind Terng's theorem? Is there a theorem which encompasses both Terng's results and the general convexity theorems for momentum maps?

In [20], a modification of the Lie–Poisson structure is used to construct connections with interesting holonomy. What are the Poisson-geometric implications of this construction?

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