Math 403 Midterm Solution
Thur 6 March 2003

Justify all of your answers using relevant terms and results from the course.

1. *Let* $E = \{1/n | n = 1, 2, 3, \ldots\}$ *be a subset of the euclidean space* $\mathbb{R}$. *Show that* 0 *is a limit point of* E *in* $\mathbb{R}$.
   We must show that, given any $r > 0$, the punctured neighborhood $N_r(0)\backslash\{0\}$ containts at least one point of E. Since $\mathbb{R}$ is archimedean, there exists an integer $n > 1/r$. Then $d(0, 1/n) = |0 - 1/n| = 1/n < r$, and so $1/n$ is the desired point of E.

2. *Define* $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *by* $d(x, y) = \sqrt{|x - y|}$ *for all* $x, y \in \mathbb{R}$. *Show that* d *is a metric.*
   We must verify that d satisfies the three defining properties of a metric, namely, for all $x, y, z \in \mathbb{R}$, we have (i) positivity, $d(x, y) \geq 0$ with equality if and only if $x = y$, (ii) symmetricity $d(x, y) = d(y, x)$, (iii) the triangle inequality, $d(x, z) \leq d(x, y) + d(y, z)$.

   Notice that $d^2(x, y) = |x - y|$ is the standard euclidean metric on $\mathbb{R}$, and so satisfies (i)–(iii). Thus, $d(x, y) = \sqrt{d^2(x, y)} \geq 0$ since square root takes non-negative value. We have equality if and only if $x = y$ since d^2 is a metric and square root only vanishes when the argument vanishes. This shows (i). For (ii), $d(x, y) = \sqrt{d^2(x, y)} = \sqrt{d^2(y, x)} = d(y, x)$. For (iii), $d^2(x, z) \leq d^2(x, y) + d^2(y, z)$ since d^2 is a metric, and $0 \leq 2d(x, y)d(y, z)$ since we have already shown positivity of d. Adding together these two inequalities gives

   $$d^2(x, z) \leq d^2(x, y) + 2d(x, y)d(y, z) + d^2(y, z) = (d(x, y) + d(y, z))^2,$$

   and since square root is an increasing function, and d is nonnegative, taking the square root of each side preserves the inequality. Thus, $d(x, z) \leq d(x, y) + d(y, z)$ which is the desired result.

3. *Show that the set of all* eventually zero *binary sequences*

   $$E = \{(x_1, x_2, x_3, \ldots) | x_i \in \{0, 1\} \textit{ and only finitely many } x_i \neq 0\}$$

   is countable.
   For each integer $n \geq 0$, let

   $$E_n = \{(x_1, x_2, \ldots) | x_j = 0 \text{ for all } j \geq n + 1, x_n = 1\}.$$

   Then $E = \bigcup_{n=0}^{\infty} E_n$ and each E_n has 2^{n-1} elements. Finally, a theorem from class guarantees that a countable union of at most countable sets is countable.

   Another proof is simply to note that E is simply the set of base 2 expansions of the nonnegative integers, and the nonnegative integers are countable.

4. *Show that a finite subset of an arbitrary metric space is compact.*
Let $E = \{x_1, \ldots, x_n\}$ be a finite subset of X, and let $\{G_\alpha\}$ be an open cover of E. For each $x_i \in E$, there exists some G_α such that $x_i \in G_\alpha$, otherwise $\{G_\alpha\}$ doesn't cover E; denote the G_α as G_i. Then

$$E = \bigcup_{i=1}^{n} \{x_i\} \subset \bigcup_{i=1}^{n} G_i,$$

showing that $\{G_1, \ldots, G_n\}$ is a finite subcover of E.

We have shown that an arbitrary open cover of E has a finite subcover, which means that E is compact.

5. *Consider the sequence* $\{1, 1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \ldots\}$*; i.e.* $x_{2n} = \frac{1}{n}$ *and* $x_{2n-1} = n$*. Find* $\limsup_{n\to\infty} x_n$ *and* $\liminf_{n\to\infty} x_n$ *and prove that your answer is correct.*
$\boxed{\limsup_{n\to\infty} x_n = +\infty}$ The limsup is the least upper bound of the set of subsequential limits. Notice that $\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} n = +\infty$. Thus, $+\infty$ is a subsequential limit. Thus, the only possible upper bound on the set of subsequential limits is $+\infty$.

$\boxed{\limsup_{n\to\infty} x_n = 0}$ The liminf is the greatest lower bound on the set of subsequential limits. Note that $\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} 1/n = 0$. Thus, 0 is a subsequential limit. We will show that 0 is the smallest subsequential limit, which suffices to show that 0 is the liminf. Given any convergent subsequence $\{x_{n_k}\} \to x$, we know that the terms of the sequence x_{n_k} are all positive; i.e. $x_{n_k} > 0$. Thus, $lim_{n\to\infty} x_{n_k} = x \geq 0$. Hence, any subsequential limit is at least 0.

For each of the following statements, indicate whether the statement is true or false. If true, give a proof. If false, give a counterexample.

6. *If a set is open, it cannot be closed.*
   False In any metric space X, the set X is open and closed.

7. *A union of closed sets is a closed set.*
   False Consider the open interval $(0, 1)$ in the metric space $\mathbb{R}$. This set is not closed since 0 and 1 are limit points of the open interval that are not contained in the open interval. On the other hand,

$$(0, 1) = \bigcup_{x \in (0,1)} \{x\},$$

   and each one point set $\{x\}$ is a closed set.

8. *Let E be a subset of a metric space X. Suppose that each point $x \in E$ has a neighborhood which contains only the point x. Then E is an open set of X.*
   True To see that E is open, we must show that every point x of E is an interior point; i.e. that there exists a neighborhood of x that is entirely contained in E. Well, by hypothesis, each point x of E has a neighborhood N consisting of only the point x. Thus, $N = \{x\} \subset E$ shows that x is interior to E.

9. *Recall that, if E is a subset of a metric space X, then E' is the set of limit points of E in X. Then $A' \cup B' = (A \cup B)'$ for any two subsets of X, A and B.*
$\boxed{\text{True}}$ The following result will be useful. Suppose that C and D are two subsets of X where $C \subset D$. Let us show that $C' \subset D'$. To see this, suppose that $p \in C'$; i.e. p is a limit point of C. Then, for any $r > 0$, the punctured neighborhood $N_r(p) \backslash \{p\}$ contains a point q of C. Since $C \subset D$, q must also be a point of D. Thus, any punctured neighborhood of p contains a point of D. But this shows that p is a limit point of D. This shows the result.

Now, since $A \subset A \cup B$, the result above shows that $A' \subset (A \cup B)'$. Similarly, $B' \subset (A \cup B)'$. Combining these two statements shows $A' \cup B' \subset (A \cup B)'$.

To show that the containment $A' \cup B' \subset (A \cup B)'$ is actually an equality, suppose that p is a point of X where $p \notin A'$ and $p \notin B'$. Then there exists some punctured neighborhood $N_{r_1}(p) \backslash \{p\}$ not containing any points of A, and another punctured neighborhood $N_{r_2}(p) \backslash \{p\}$ not containing any points of B. Setting $r = \min\{r_1, r_2\}$ we know that the punctured neighborhood $N_r(p) \backslash \{p\}$ doesn't contain any points of A or B; i.e. no points of $A \cup B$. This shows that p is not a limit point of $A \cup B$.

10. *Any Cauchy sequence is a convergent sequence.*
$\boxed{\text{False}}$ Consider the open interval $Y = (0, 1)$ as a subspace of the euclidean space $\mathbb{R}$. Let us show that $\{1/n | n = 1, 2, \ldots\}$ is a Cauchy sequence of Y that is not a convergent sequence of Y. We saw in class that, if Y is a subspace of X, then any Cauchy sequence of X taking values in Y is also a Cauchy sequence of Y. We also saw that all convergent sequences are Cauchy.

Now, $\{1/n\} \to 0$ is a convergent sequence in $\mathbb{R}$, and so is a Cauchy sequence of $\mathbb{R}$. Since the sequence takes values in the open interval $Y = (0, 1)$, it is a Cauchy sequence in Y.

To see that it doesn't converge in Y, suppose to the contrary that $\{1/n\} \to y$ where $y \in Y$. Then $\{1/n\} \to y$ where $y \in X$. Since we know that a convergent sequence has a unique limit, $y = 0$, which is a contradiction since $0 \notin Y$.