Math 27: Calculus I
Demonstration Midterm 3 Solutions

Justify your answers using relevant terms and results from the course.

1. *Show that* $g(t) = \sqrt{t} + \sqrt{1+t} - 4$ *has exactly one zero on the interval* $(0, \infty)$.

   First we will show that $g(t) = 0$ has at least one solution. To see this, note that
   $$\begin{aligned} g(\text{small}) &= \sqrt{\text{small}} + \sqrt{1 + \text{small}} - 4 \\ &= \text{small} + 1 + \text{small} - 4 \end{aligned}$$
   and so $g(\text{small}) =$ is negatively valued. Also,
   $$\begin{aligned} g(\text{BIG}) &= \sqrt{\text{BIG}} + \sqrt{1 + \text{BIG}} - 4 \\ &= \text{BIG} + \text{BIG} - 4 \\ &= \text{BIG} \end{aligned}$$
   and so $g(\text{BIG})$ is positively valued. Moreover, we know that $g(t)$ is a continuous function since it is an algebraic combination of continuous functions. Therefore, the Intermediate Value Theorem guarantees that $g(c) = 0$ for some $t = c$ between small and BIG.

   In fact, $t = c$ is the only zero of $g(t)$. To show this, we will verify that $g(t)$ is an increasing function. By the first derivative test, it suffices to show that the derivative $g'(t)$ is positively valued. Compute that
   $$g'(t) = \frac{1}{\sqrt{t}} + \frac{1}{\sqrt{1+t}}$$
   and note that both terms are positively valued.

2. *Suppose that a body has a constant acceleration of* $a = 32\,ft/sec^2$. *Find a formula for the body's position* s *at time* t *if the velocity* v *at time* $t = 0$ *is given by* $v(0) = 20\,ft/sec$ *and the position* s *at time* $t = 0$ *is given by* $s(0) = 5$.

   First we will find a formula for the velocity, $v(t)$. We know that $\frac{dv}{dt} = a = 32$ by definition of acceleration. Integrating, we find that
   $$v = \int \frac{dv}{dt}\,dt = \int 32\,dt = 32t + C.$$

Substituting the initial condition $v(0) = 20$ we find that

$$32 \cdot 0 + C = 20 \Rightarrow C = 20.$$

Hence, the exact formula for velocity is $\boxed{v(t) = 32t + 20.}$

Next, we will find a formula for position, $s(t)$. We know $\frac{ds}{dt} = v = 32t + 20$. Integrating, we find that

$$s = \int \frac{ds}{dt}\, dt = \int 32t + 20\, dt = 16t^2 + 20t + C.$$

Substituting the initial condision $s(0) = 5$ we find that

$$16 \cdot 0^2 + 20 \cdot 0 + C = 5 \Rightarrow C = 5.$$

Hence, the exact formula for position is $\boxed{s(t) = 16t^2 + 20t + C.}$

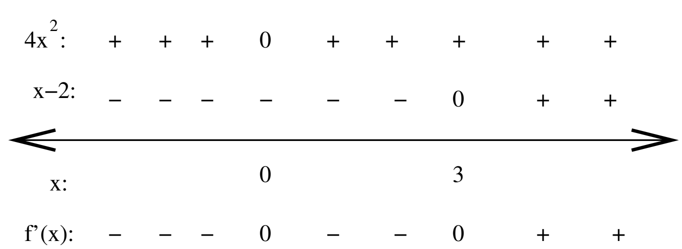
3. *Sketch the graph of the function* $f(x) = x^4 - 4x^3 + 10$. *Indicate precisely where* f *is increasing, decreasing, bending up, bending down, and the location of all critical and inflection points.*

Where is $f(x) = +/0/-$? No idea. Move on.

Where is $f'(x) = +/0/-$? Compute the derivative

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3).$$

Then determine the signs of $f'(x)$ using the number-line technique illustrated below:



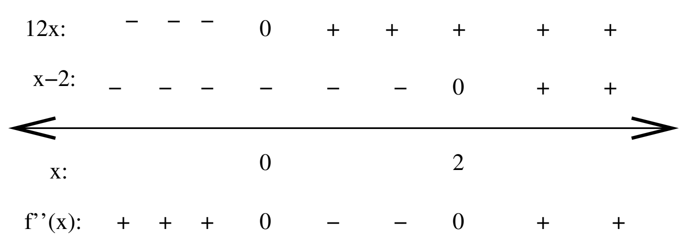
Thus, the critical points are located at

$$(x, y) = (0, 10) \text{ and } (x, y) = (3, -17).$$

Where is $f''(x) = +/0/-$? Compute the second derivative

$$f''(x) = 12x^2 - 24x = 12x(x-2).$$

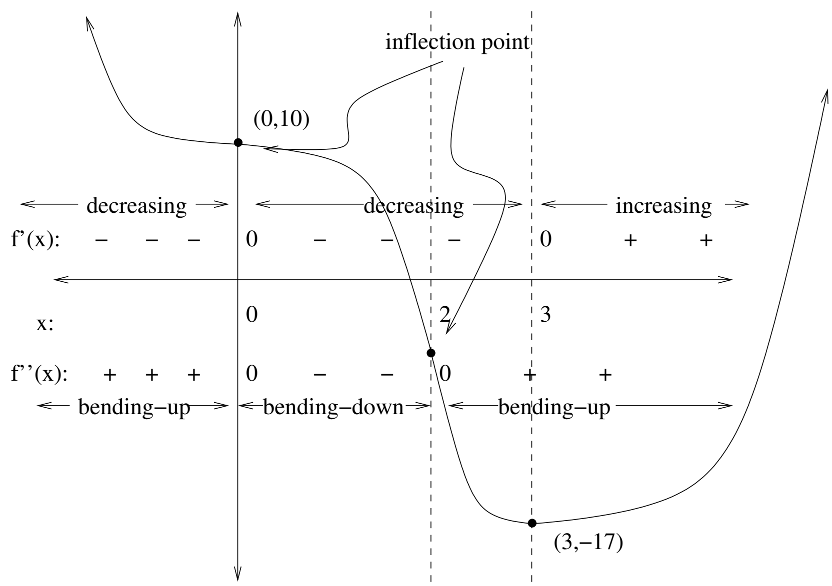
Then determine the signs of $f''(x)$ using the number-line technique illustrated below:



Thus, the inflection points are located at

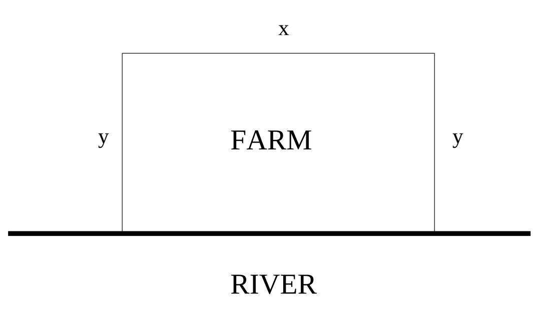
$$(x, y) = (0, 10) \text{ and } (x, y) = (2, -6).$$

We now sketch the graph of $f(x)$ by plotting the critical and inflections points and applying the first and second derivative tests.



4. *A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 meters of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?*

Let x denote the length of the side of the fence parallel to the river, and y denoted the length of each of the sides perpendicular to the river, as in the diagram below.



Since the fence length is 800 meters,

$$x + 2y = 800 \Rightarrow x = 800 - 2y.$$

The area of the farm enclosed is

$$A = xy = (800 - 2y)y = 800y - 2y^2.$$

To find the maximum area, we solve the equation $A'(y) = 0$. Well,

$$A'(y) = 800 - 4y,$$

thus $y = 200$ solves the equation $A'(y) = 0$. Substituting into the formula $x = 800 - 2y$, we find that $x = 400$. Thus, the enclosed area is $A = 400 \cdot 200 = 80,000\, m^2$.

5. *Use l'Hospital's rule to evaluate the limit*

$$\lim_{x \to 0} \frac{2x}{x + 7\sqrt{x}}.$$

Evaluation by substitution yields the indeterminate form 0/0 so we may directly apply l'Hospital's rule:

$$\lim_{x \to 0} \frac{2x}{x + 7\sqrt{x}} = \lim_{x \to 0} \frac{2}{1 + \frac{7}{\sqrt{x}}}.$$

Evaluation by substitution of the limit on the right side yields $\frac{2}{1+\frac{7}{\sqrt{0}}}$ which is undefined. Use the BIG/small method to evaluate this limit:

$$\begin{aligned} \lim_{x \to 0} \tfrac{2}{1+\frac{7}{\sqrt{x}}} &= \lim_{x \to 0} \tfrac{2}{1+\frac{7}{\sqrt{\text{small}}}} \\ &= \tfrac{2}{1+\frac{7}{\text{small}}} \\ &= \tfrac{2}{1+\text{BIG}} \\ &= \tfrac{2}{\text{BIG}} \\ &= \text{small} \\ &= 0. \end{aligned}$$

Thus, the required limit is $\boxed{0}$.

6. *Solve the following problems on* Σ*-summation notation:*

(a) *Evaluate the sum* $\displaystyle\sum_{i=0}^{3} \sin\left(\frac{i\pi}{6}\right)$.

Expand and evaluate the sum:

$$\begin{aligned}\sum_{i=0}^{3} \sin\left(\frac{i\pi}{6}\right) &= \sin(0) + \sin\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{2}\right) \\ &= 0 + \frac{1}{2} + \frac{\sqrt{3}}{2} + 1 \\ &= \frac{3}{2} + \frac{\sqrt{3}}{2}\end{aligned}$$

(b) *Write the following sum in* Σ*-summation notation:*

$$\frac{7}{3} + \frac{7}{5} + \frac{7}{7} + \frac{7}{9}.$$

The denominators are the second, third, forth, and fifth odd numbers. A formula for the i^{th} odd number is $2i - 1$. Thus,

$$\frac{7}{3} + \frac{7}{5} + \frac{7}{7} + \frac{7}{9} = \sum_{i=2}^{5} \frac{7}{2i-1}.$$

(c) *Evaluate the sum* $\displaystyle\sum_{i=1}^{10} 2i^2 - 3i + 6$.

We proceed by breaking up the sum and then applying the summation formulas:

$$\begin{aligned}\sum_{i=1}^{10} 2i^2 - 3i + 6 &= \sum_{i=1}^{10} 2i^2 - \sum_{i=1}^{10} 3i + \sum_{i=1}^{10} 6 \text{ by the sum law} \\ &= 2\left(\sum_{i=1}^{10} i^2\right) - 3\left(\sum_{i=1}^{10} i\right) + 6\left(\sum_{i=1}^{10} 1\right) \\ &\quad \text{by the constant multiple rule} \\ &= 2\left(\frac{(10)(11)(21)}{6}\right) - 3\left(\frac{(10)(11)}{2}\right) + 6(10) \\ &= 10 \cdot 11 \cdot 7 - 3 \cdot 55 + 60 \\ &= 665.\end{aligned}$$