Math 185
Assignment 12
Due Wednesday Dec 6

1. Evaluate the following integrals by applying the Residue Theorem as indicated in lecture of Nov 22.

   (a) $\int_{\infty}^{\infty} \frac{dx}{x^2-2x+4}$

   (b) $\int_{0}^{\infty} \frac{dx}{1+x^2}$

   (c) $\int_{0}^{\infty} \frac{x^{a-1}}{1+x^3}\,dx$ for $0 \le a \le 3$

2. Extend the zero-pole counting theorem $\int_C \frac{df}{f} = Z - P$ to include the following result: If f is analytic on A except for zeros at $a_1, \ldots, a_n$ and poles at $b_1, \ldots, b_m$ (each repeated according to its multiplicity), and h is analytic on A, and C is a simple closed curve in A with interior in A passing through none of $a_1, \ldots, a_n, b_1, \ldots, b_m$, then

$$\int_C \frac{f'(z)}{f(z)} h(z)\,dz = 2\pi i \left[\sum_{k=1}^{n} h(a_k) - \sum_{k=1}^{m} h(b_k) \right].$$

3. Use the previous exercise to prove that, if $f(z)$ is polynomial then

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} z\,dz$$

   is the sum of the zeros of $f(z)$ if the circle C is large enough.

4. Suppose $g_n(z) = \sum_{k=0}^{n} 1/(k!z^k)$, and let $\epsilon > 0$. For large enough n, are all the zeros of g_n inside the circle C_ϵ of radius ϵ centered at 0?

   Hint: For large values of n, $g_n(\xi) \approx e^{-\xi}$ for $\xi \in C_\epsilon$.

5. In this problem we determine the number of roots of $f(z) = z^7 - 2z^5 + 6z^3 - z + 1$ are contained in various annuli centered at the origin.

   (a) Let C_1 be the unit circle. Set $g(z) = 6z^3$. Use Rouché's Theorem to show that $g(z)$ and $f(z)$ have the same number of roots (repeatedly counting each root according to its multiplicity) inside C_1. How many roots does $f(z)$ have inside C_1?

   (b) Let C_2 be the circle of radius 2. Set $g(z) = z^7$. Use Rouché's Theorem to show that $g(z)$ and $f(z)$ have the same number of roots inside C_2. How many roots does $f(z)$ have inside C_2?

   (c) Let $A = \{1 < |z| < 2 | z \in \mathbb{C}\}$ be an annulus. Combining (a) and (b) above, how many roots does $f(z)$ have inside A?

6. In this problem we show that if $f : \mathbb{C} \to \mathbb{C}$ is entire and one-to-one, then $f(z) = az + b$.

   (a) Suppose that $f(z)$ is a polynomial. Use the fundamental theorem of algebra to show that f being one-to-one forces f to be linear.

   (b) Suppose that $f(z)$ is trancendental. We will show this leads to a contradiction as follows.

      i. Prove that there must be some $z_0 \in \mathbb{C}$ such that $f'(z_0) \neq 0$.

      ii. Since $f'(z_0) \neq 0$, we may apply the inverse function theorem, which states that there exists a neighborhood U of z_0 and a neighborhood V of $w_0 = f(z_0)$ such that $f : U \to V$ is one-to-one and onto. Using the fact that f is one-to-one on all of $\mathbb{C}$, show that if $f(z) \in V$ then $z \in U$.

      iii. Use Casorati-Weierstrass to show that there exists a sequence $\{z_n\} \to \infty$ where $\{f(z_n)\} \to w_0$.

      iv. Explain why, for large enough n, $f(z_n) \in V$.

      v. Combine parts iv. and ii. to show that, for large enough n, $z_n \in U$.

      vi. Observe that the conclusion of v. contradicts $\{z_n\} \to \infty$ from part iii.