Math 160: History of Mathematics
Assignment 6
Due Wed Apr 13

Your solutions should be written so-as to be clear to an audience of fellow math 160 students.

Use the Diophantine method of adequating to find two non-negative rational numbers summing to one, such that, if 12 is added to either, the result is a square (of a rational number). You may wish to follow the steps below.

It suffices to find rational numbers x and y such that

$$\begin{aligned} x^2 + y^2 &= 25 \\ -1 \leq x^2 - y^2 &\leq 1 \end{aligned}$$

since then $x^2 - 12$ and $y^2 - 12$ are the required non-negative rational numbers. We do this as follows.

1. Show that if $P = (3,4)$ and $Q = (a,b)$ has rational coordinates, then the line from P to Q intersects the circle $x^2 + y^2 = 25$ in another point $N = (x,y)$ also having rational coordinates.

2. The point $M = (\sqrt{\frac{25}{2}}, \sqrt{\frac{25}{2}})$ lies on the circle $x^2 + y^2 = 25$. Explain why the coordinates of M are irrational.

3. Use the method below to find that $\frac{99}{28}$ is a good rational approximation of $\sqrt{\frac{25}{2}}$.

   (a) Let rational numbers α (small) and β solve the equation $\frac{25}{2} + \alpha^2 = \beta^2$. Explain why β is the desired approximation to $\sqrt{\frac{25}{2}}$.

   (b) Multiply through by 4 to get $50 + (2\alpha)^2 = (2\beta)^2$.

   (c) Divide by $(2\alpha)^2$ to get $50(\frac{1}{2\alpha})^2 + 1 = (\frac{\beta}{\alpha})^2$.

   (d) Make the variable substitution $x = \frac{1}{2\alpha}$ and $7x + 1 = \frac{\beta}{\alpha}$ into equation (c). Solve for x.

   (e) Use the x value from part (d) to determine the values of α and β.

4. Let $P = (3,4)$ and $Q = (\frac{99}{28}, \frac{99}{28})$. Show that the parametrized line through P and Q given by $(x_t, y_t) = (1-t)P + tQ$ is also given by the formulas

$$\begin{aligned} x_t &= 3 + \frac{15}{28}t \\ y_t &= 4 - \frac{13}{28}t \end{aligned}$$

5. The line PQ in 4 above intersects the circle $x^2 + y^2 = 25$ in two points: $P = (3,4)$ and $N = (x_t, y_y)$ for some value of t. Determine the coordinates of N as follows.

   (a) Substitute the formulas for x_t and y_t from part 4 into the equation of the circle $x^2 + y^2 = 25$ and solve for t to obtain $t = \frac{392}{394}$.

   (b) Substitute the t-value from (a) into the formulas for x_t and y_t from part 4 to obtain

$$\begin{aligned} x_t &= \frac{696}{197} \\ y_t &= \frac{697}{197} \end{aligned}$$

6. Using the x_t- and y_t-values from part 5, compute $x_t^2 - 12$ and $y_t^2 - 12$ and conclude that $\frac{20101}{38809}$ and $\frac{18708}{38809}$ are the required non-negative rational numbers solving the Diophantine problem.

7. Verify that the values in part 6. indeed solve the Diophantine problem, i.e., that they sum to 1 and that if 12 is added to either of them, the result is a square (of a rational number).