Math 1: Fundamental Mathematical Concepts I
Study Guide Solutions for Midterm 2

Midterm 2 will feature a selection of these types of problems

Justify your answers using relevant terms and results from the course.

1. *Find the base ten equivalent of each of the following.*

   (a) $101101_{\text{two}} = 1 \cdot 32 + 0 \cdot 16 + 1 \cdot 8 + 1 \cdot 4 + 0 \cdot 2 + 1 \cdot 1 = 45_{\text{ten}}$.

   (b) $123_{\text{five}} = 1 \cdot 25 + 2 \cdot 5 + 3 \cdot 1 = 38_{\text{ten}}$.

   (c) $346_{\text{seven}} = 3 \cdot 49 + 4 \cdot 7 + 6 \cdot 1 = 181_{\text{ten}}$.

2. *Write 287_{ten} as a numeral in each base indicated.*

   (a) 100011111_{two}

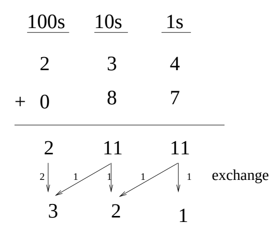
   (b) 2122_{five}

   (c) 560_{seven}

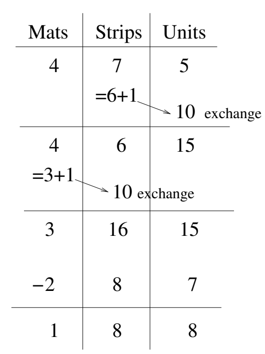
3. *Compute the sum $47 + 25$ using the mats, strips, units representation and indicating any necessary exchanges.*

	Mats	Strips	Units
	0	4	7
+	0	2	5
	0	6	12
			=10+2
		1 ← exchange	
	0	7	2

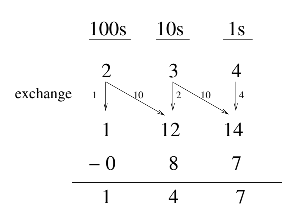
4. *Compute the sum* $234 + 87$ *using place-value diagrams.*



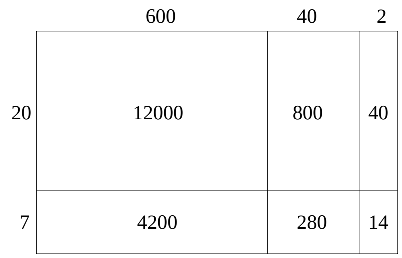
5. *Compute the difference* $475 - 287$ *using the mats, strips, units representation and indicating any necessary exchanges.*



6. *Compute the difference* $234 - 87$ *using place value diagrams.*



7. *Compute the product* 642×27 *using the rectangular array algorithm.*

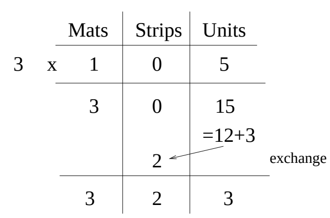


Thus,

$$642 \times 27 = 12000 + 4200 + 800 + 280 + 40 + 14 = 17334.$$

8. *Use a base 6 mats,strips, units representation to compute the product* $3_{six} \times 105_{six}$. *Make any exchanges necessary to write the final answer in base 6 notation.*

   Since we are working in base 6, each unit represents 1, each strip is made of 6 units, and each mat is made of $6 \times 6 = 36$ units.



9. *Perform the following calculations in base five notation using the standard addition, subtraction, and multiplication algorithms. Assume that the numerals are already written in base five.*

10. *Round 274,535 to the nearest*

    (a) *thousand.* 275,000

    (b) *ten thousand.* 270,000

    (c) *one hundred thousand.* 300,000

(a) $\begin{array}{r} {}^{1\,1} \\ 2433 \\ +141 \\ \hline 3124 \end{array}$ (b) $\begin{array}{r} 3 \\ 2433 \\ -141 \\ \hline 2242 \end{array}$ (a) $\begin{array}{r} {}^{3\,2} \\ {}^{1\,1}243 \\ \text{x}42 \\ \hline 1041 \\ +21320 \\ \hline 22411 \end{array}$

11. *The states of Otah, Molorado, New Lexico, and Garizona have areas 82,168 square miles, 153,730 square miles, 141,365 square miles, and 183,642 square miles. Explain how to use rounding to estimate the combined area of the four states.*

    Start by rounding each state's area in the left-most digit: The approximate areas are given by Otah $\approx 80,000\text{mi}^2$, Molorado $\approx 200,000\text{mi}^2$, New Lexico $\approx 100,000\text{mi}^2$, and Garizona $\approx 200,000\text{mi}^2$. Summing the approximations gives the combined area as approximately

$$80,000 + 200,000 + 100,000 + 200,000 = 580,000\text{mi}^2$$

12. *Demonstrate that 97 is prime.*

    If 97 is not prime then there must be a prime divisor less than $\sqrt{97} = 9.84\ldots$. Thus, we need only verify that 97 is not divisible by $2, 3, 5$ or 7. Indeed, 97 is not even, it is not divisible by 3 since $9 + 7 = 16$ which is not divisible by 3, it is not a multiple of 5 since the 1s digit 7 is not 0 or 5, and finally $97 \div 7$ has remainder 6 by long division.

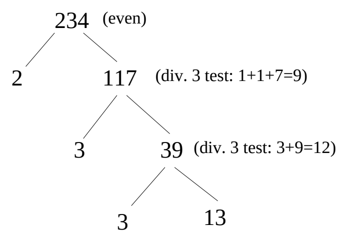
13. *Use Euclid's prime trick to produce a prime different than 2, 3 and 11.*

    Euclid's prime trick requires that we compute the prime factorization of $m = 2 \cdot 3 \cdot 11 + 1 = 67$. But 67 is prime so we are done.

14. *Is there a largest prime number? Justify your answer.*

    No, there is no largest prime number. If there we are largest prime number, then there would be only finitely many primes. But Euclid's prime trick allows us to create a new prime not contained in any given finite set of primes. Thus, the set of all primes cannot be finite.

15. (a) *Determine the prime factorization of 234.*



(b) *Use the Prime Factorization Divisibilility Test to determine all divisors of 234.*

Let m be any divisor of 234. By the PFDT each prime p appears at least as many times in the prime factorization of 234 as it does in the prime factorization of m. Thus, the prime factorization of m can have at most one 2, at most two 3s, and at most one 13. In particular, $m = 2 \cdot 3^2 \cdot 13 = 234$, $3^2 \cdot 13 = 117$, $2 \cdot 3 \cdot 13 = 78$, $2 \cdot 3^2 = 18$, $3 \cdot 13 = 39$, $3^2 = 9$, $2 \cdot 3 = 6$, $2 \cdot 13 = 26$, 2, 3, 13, 1.

Optionally, we may double check this. Observe that there are 2 possibilities for the divisor 2 (one 2 or zero 2s), 3 possibilities for the divisor 3 (two 3s, one 3, or zero 3s), and 2 possibilities for the divisor 13 (one 13 or zero 13s). Thus, there are $2 \cdot 3 \cdot 2 = 12$ total possibilities for the divisor m. Indeed, there are twelve divisors on our list above, so it must be complete.

16. (a) *Write the division equation for the division problem* $7898 \div 55$.

Long division shows that 7898 divided by 55 has quotient 143 and remainder 33. Plugging these values into the division equation $a = qb + r$ gives $7898 = 143 \cdot 55 + 33$.

(b) *Suppose a is an arbitrary 4 digit whole number where the remainder r of* $a \div 55$ *is divisible by 11. Must a itself be divisible by 11? Justify your answer.*

Yes, a must be divisible by 11. To see this, write the division equation $a = q \cdot 55 + r$ Since 55 is divisible by 11, $q \cdot 55 = (q \cdot 5) \cdot 11$ is divisible by 11 also. Moreover, we are told that r is divisible by 11. Thus, by the divisibility of sums rule, $q \cdot 55 + r$ is divisible by 11. But $q \cdot 55 + r = a$ by the division equation.

17. *Find GCD*$(2^5 3^4 5^3 7^2 11^1 13^0, 2^0 3^1 5^2 7^3 11^4 13^5)$. *Justify your answer.*

    By the Prime Factorization Divisibility Test (or by Workshop 3, or by a theorem from chapter 4), each prime in the factorization of the GCD has exponent given by the lesser of the exponents of the corresponding primes in the factorizations of each of the given numbers. Thus,

$$\text{GCD} = 2^0 3^1 5^2 7^2 11^1 13^0$$

18. *Find GCD*$(132, 209)$ *using the Euclidean algorithm.*

    The Euclidean algorithm tells us to repeatedly replace the larger number with the remainder on division by the smaller number. In this case, we get the following:

$$\begin{aligned} \text{GCD}(132, 209) &= \text{GCD}(132, 77) \\ &= \text{GCD}(77, 55) \\ &= \text{GCD}(55, 22) \\ &= \text{GCD}(22, 11) \\ &= \text{GCD}(11, 0) \\ &= 11 \end{aligned}$$

19. *Fill in the missing digit so that 43872_27834 is divisible by 9. Justify your answer using a divisibility test.*

    The divisibility by 9 test requires that the sum of the digits be divisible by 9. Aside from the missing digit, the sum of the given digits is $4+3+8+7+2+2+7+8+3+4 = 48$. Since $48+6 = 54$ the missing digit must be 6.

20. *Suppose a whole number a is divisible by 2 and also by 3. Is a necessarily divisible by 6? Explain.*

    Yes, a must be divisible by 6. Since a is div. by 2 and by 3, these primes must both appear in the prime factorization of a by the prime factorization divisibility test (PFDT). Since $6 = 2 \cdot 3$ and the primes 2 and 3 both appear in the prime factorization of a, a is div. by 6 by the PFDT.

21. *Suppose a whole number a is divisible by 6 and by 10. Is a necessarily divisible by 60? Explain.*

    No, for example $a = 30$ is divisible by 6 and by 10, but 30 is not div. by 60. The argument in the previous problem fails here because 6 and 10 are not prime.