Jan 18: Math Goes to Hollywood
Workshop 5
Due Thursday Jan 25

Justify your answers using relevant terms and concepts from the course.

1. Let the *integers extended by* $\sqrt{3}$ be all real numbers of the form $a+b\sqrt{3}$ where a and b are themselves integers. For example $-7+5\sqrt{3}$ is such an extended integer.

   (a) Show that the Fermat equation

   $$a^2 - 3b^2 = \pm 1$$

   factors in these extended integers.

   (b) Check that $a = 2$ and $b = 1$ is a solution to the Fermat equation in part (a).

   (c) Compute the first five powers of $2 + 1\sqrt{3}$,

   $$(2+1\sqrt{3})^1,\ (2+1\sqrt{3})^2,\ (2+1\sqrt{3})^3,\ (2+1\sqrt{3})^4,\ (2+1\sqrt{3})^5$$

   and check that a and b values of each power give solutions to the Fermat equation.

   (d) Recall that an extended integer solution $a + b\sqrt{3}$ to the Fermat equation corresponds to the rational approximation of $\sqrt{3}$ given by a/b. Find the fractions corresponding to the extended integer solutions you found in part (c).

   (e) Suppose that $a+b\sqrt{3}$ is an extended integer solution of the Fermat equation. Explicitly compute the product

   $$(a + b\sqrt{3})(2 + 1\sqrt{3}).$$

   (f) Write the fractions corresponding to the extended integer $a+b\sqrt{3}$ and the product you computed in part (e). [*This should yield the same rule to obtain rational approximations of* $\sqrt{3}$ *we proved in problem 4 of workshop 3. Amazing!*]

2. Redo problem 1 for the *integers extended by* $\sqrt{5}$ being sure to replace the number "3" with the number "5" throughout.

3. The slick part of Fermat's method is that, given one solution to the Fermat equation, we can always compute more by using the extended integers. But it can be difficult to find that initial solution to the Fermat equation. For example, consider the Fermat equation

$$a^2 - 7b^2 = \pm 1.$$

   (a) There is a solution where a and b are both less than 10. Find it.

   (b) Redo the steps of problem 1 for the *integers extended by* $\sqrt{7}$ using the initial solution to the Fermat equation you found in part (a).