Math 112
Assignment 8
Due Wed Apr 21

1. **Stewart 8.9: Solve the general cubic by Ruffini radicals.** Follow the steps below.

   (a) Explicitly write the elementary symmetric polynomials s_1, s_2, s_3 in the variables t_1, t_2, and t_3.

   (b) Recall the polynomial $\delta = (t_1 - t_2)(t_1 - t_2)(t_2 - t_3)$. Explain why δ^2 is a symmetric polynomial.

   (c) According to the fundamental theorem of symmetric polynomials, eve(ry symmetric polynomial can be expressed in terms of the **elementary** symmetric polynomials. Find a formula for δ^2 as a polynomial in s_1, s_2 and s_3, and then take the square root to find a formula for δ.

   (d) Conclude from part (c) that $\delta^2 \in \mathbb{C}(s_1, s_2, s_3)$, and thus that

   $$\mathbb{C}(s_1, s_2, s_3)(\delta) : \mathbb{C}(s_1, s_2, s_3)$$

   is a radical extension.

   (e) The alternating group A_3 consists of 3 permutations. Write them explicitly.

   (f) Define the variables β_1, β_2 and β_3 by the formulas

   $$\begin{aligned} \beta_1 &= t_1 + t_2 + t_3 \\ \beta_2 &= t_1 + \zeta t_2 + \zeta^2 t_3 \\ \beta_3 &= t_1 + \zeta^2 t_2 + \zeta t_3 \end{aligned}$$

   where $\zeta = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ is a third root of unity. Show that for each permutation $\sigma \in A_3$,

   $$\sigma\beta_j = \zeta^k \beta_j$$

   for some $k = 0, 1$, or 2. Conclude that each β_j^3 is fixed by the alternating group A_3.

   (g) Recall from our lemma in class that the fixed field of A_3 is $\mathbb{C}(s_1, s_2, s_3)(\delta)$. Combine this with your conclusion in part (f) to show that each $\beta_j^3 \in \mathbb{C}(s_1, s_2, s_3)(\delta)$. Write each β_j^3 explicitly as a polynomial in s_1, s_2, s_3, and δ. Then take cube roots to find formulas for each β_j. *Actually, the formula for β_1 can be seen immediately without finding β_1^3; what is it?*

   (h) Explain why the relatively simple formula for β_1 in part (g) shows that

   $$\mathbb{C}(s_1, s_2, s_3, \delta, \beta_1, \beta_2, \beta_3) = \mathbb{C}(s_1, s_2, s_3, \delta, \beta_2, \beta_3)$$

(i) Conclude from part (g) that both

$$\mathbb{C}(s_1, s_2, s_3, \delta)(\beta_2) : \mathbb{C}(s_1, s_2, s_3, \delta)$$

$$\mathbb{C}(s_1, s_2, s_3, \delta, \beta_2)(\beta_3) : \mathbb{C}(s_1, s_2, s_3, \delta, \beta_2)$$

are radical extensions.

(j) Use linear algebra to solve the system of equations in part (f) for the variables t_1, t_2, and t_3.

(k) Explain why the formulas in part (j) solve the general cubic (i.e. they tell us the roots of any cubic in terms of the coefficients).

(l) Use parts (h) and (j) that

$$\mathbb{C}(t_1, t_2, t_3) = \mathbb{C}(s_1, s_2, s_3, \delta, \beta_2, \beta_3)$$

(m) Combine (d), (g), (i), and (l) to show that we have solved the cubic by Ruffini radicals.