# Math 112
# Assignment 1
# Due Wed Feb 20

1. Recall that *complex conjugation* is a map of complex numbers $\bar{\ }: \mathbb{C} \to \mathbb{C}$ given by $\overline{a+ib} = a - ib$.

   (a) Show that complex conjugation is an automorphism of $\mathbb{C}$.

   (b) Suppose that $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ is a polynomial with real coefficients $a_0, a_1, \ldots, a_n$. Using part (a), show that $p(\overline{z}) = \overline{p(z)}$ for all $z \in \mathbb{C}$.

2. Stewart exercise 1.2

3. Let $\alpha = \sqrt[3]{2} \in \mathbb{R}$ and let

$$K = \{p + q\alpha + r\alpha^2 | p, q, r \in \mathbb{Q}\}$$

   Show that $K$ is a subfield of $\mathbb{C}$ as follows.

   (a) Show that $K$ is a subring of $\mathbb{C}$.

   (b) Having proved part (a), to show that $K$ is a subfield of $\mathbb{C}$ it is enough to prove that the reciprocal of every element of $K$ is also in $K$. To do this, let $p + q\alpha + r\alpha^2 \in K$ be given (i.e., suppose we are provided with the rational numbers $p, q, r$). We wish to find $x + y\alpha + z\alpha^2 \in K$ such that

$$\frac{1}{p + q\alpha + r\alpha^2} = x + y\alpha + z\alpha^2$$

   or, equivalently that

$$1 + 0\alpha + 0\alpha^2 = (p + q\alpha + r\alpha^2)(x + y\alpha + z\alpha^2)$$

   Expand the product on the right-hand side and collect powers of $\alpha$ to obtain three linear equations in the variables $x, y, z$. Be sure to use the fact that $\alpha^3 = 2$.

   (c) Now solve the linear equations in part (b).

   (d) Find the reciprocal $1/\beta$ of $\beta = 1 + 2\alpha + 3\alpha^2$ using your formulas in part (c). Does it work? I.e. does your computed reciprocal $1/\beta$ multiply with $\beta$ to yield 1?

4. Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and let

$$L = \{p + q\omega\alpha + r\omega^2\alpha^2 | p, q, r \in \mathbb{Q}\}$$

   Show that $L$ is a subfield of $\mathbb{C}$ as follows.

   (a) Show that $\omega^3 = 1$.

(b) Using the technique of problem 3, show that $L$ is a subfield of $\mathbb{C}$

5. Show that the map $\phi : K \to L$ given by

$$\phi(p + q\alpha + r\alpha^2) = p + q\omega\alpha + r\omega^2\alpha^2$$

is an isomorphism.

6. Stewart exercise 2.9

7. *Here we prove the fundamental theorem of algebra for real cubic polynomials.* Suppose $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ where $a_0, \dots, a_3 \in \mathbb{R}$. Assume for simplicity that $a_3 > 0$. We wish to show that $p(x)$ has at least one real root $\alpha \in \mathbb{R}$.

   (a) Show that for sufficiently large positive values of $x$, $p(x)$ must have positive value.*[Hint: What is the value of* $\lim_{x\to\infty} \frac{p(x)}{x^3}$ *?]*

   (b) Show that for sufficiently large (how large?) negative values of $x$, $p(x)$ must have negative value.

   (c) Use the results of parts (a) and (b) with the Intermediate Value Theorem to show that $p(x)$ must have a real root $\alpha$.