Math 10: The Art and Practice of Mathematics
Ch 5.1 A: Price Competition

We summarize our example of how to find the Nash equilibrium of prices given certain sales and profit formulas for two competing firms.

Suppose that firms X and Y sell product at prices p_x and p_y, respectively, that their yearly sales q_x and q_y are given by the formulas

$$\begin{aligned} q_x &= 44 - 2p_x + p_y, \\ q_y &= 44 - 2p_y + p_x. \end{aligned}$$

If firm X has a production cost of \$6/unit, and firm Y has a production cost of \$8/unit, then the yearly profits are given by the formulas

$$\begin{aligned} b_x &= (p_x - 6)q_x, \\ b_y &= (p_y - 8)q_y. \end{aligned}$$

Example 1: *Suppose that firm X charges \$15/unit and firm Y charges \$22/unit. Determine the quantities sold and profits of each firm.*

Solution: Substitute the values $p_x = 15$ and $p_y = 22$ into the given quantity formulas to obtain

$$\begin{aligned} q_x &= 44 - 2(15) + 22 = 36, \\ q_y &= 44 - 2(22) + 15 = 15. \end{aligned}$$

Then substitute these price and quantity values into the profit formulas to obtain

$$\begin{aligned} b_x &= (15 - 6)(36) = 324, \\ b_y &= (22 - 8)(15) = 210. \end{aligned}$$

□

Exercise 1: *Suppose that firm X charges \$21/unit and firm Y charges \$17/unit. Determine the quantities sold and profits of each firm.*

Given a specific price announced by one firm, we can determine the profit maximizing best reply price for the other firm.

Example 2: *Suppose that firm Y charges \$22/unit. Determine the best reply price p_x for firm X. Then verify that firm X prefers price p_x to the price of \$15/unit as in Example 1.*

Solution: The best reply price for firm X is the price p_x that maximizes profit b_x. To find this, first substitute the value $p_y = 22$ into the quantity formula for firm X to obtain $q_x = 66 - 2p_x$. Then substitute q_x into the profit formula for firm X to obtain

$$b_x = (p_x - 6)(66 - 2p_x).$$

Next, expand the b_x-profit formula by FOILing to obtain

$$\begin{aligned} b_x &= 66p_x - 2p_x^2 - 396 + 12p_x \\ &= -2p_x^2 + 78p_x - 396. \end{aligned}$$

Since this is a quadratic equation, we can determine the maximum b_x-value by using the critical point formula as follows. Recall that, for a general quadratic equation $y = ax^2 + bx + c$, the *critical point* is given by

$$\boxed{x = -\tfrac{b}{2a}}$$

and the *critical value* is found by substituting the critical point into the given quadratic. For the quadratic b_x-profit formula above, we have $a = -2$ and $b = 78$. Thus, the critical point price is

$$p_x = -\frac{78}{2 \cdot (-2)} = 19.5,$$

and the critical value profit is

$$b_x = -2(19.5)^2 + 78(19.5) - 396 = 364.4.$$

Conclusion: The best reply price to p_y =\$22 is p_x = \$19.50 yielding profit of b_x = \$364.50. Firm X prefers this to the profit of \$324 when charging \$15/unit as in Example 1. □

Exercise 2: *Suppose that firm Y charges \$17/unit. Determine the best reply price p_x for firm X. Then verify that firm X prefers price p_x to the price of \$21/unit as in Exercise 1.*

Each time firm Y makes a new price announcement p_y we could repeat the method of Example 2 to find a best reply price p_x for firm X. Fortunately, there is a short-cut. As we now show, there is a *best reply formula* for firm X that instantly computes the best reply price p_x to any given price p_y set

by firm Y.

Example 3: *Determine the best reply formula for firm X.*
Solution: The best reply price for firm X is the price p_x that maximizes profit b_x. To find this, suppose firm Y announces a price p_y. Substitute the quantity formula $q_x = 44 - 2p_x + p_y$ into the profit formula $b_x = (p_x - 6)q_x$ for firm X to obtain

$$b_x = (p_x - 6)(44 - 2p_x + p_y).$$

Next, expand the profit b_x formula (unfortunately FOILing doesn't work since one term is not a binomial) as follows:

$$\begin{aligned} b_x &= 44p_x - 2p_x^2 + p_x p_y - 6 \cdot 44 + 12p_x - 6p_y \\ &= -2p_x^2 + (56 + p_y)p_x - 6p_y - 264 \end{aligned}$$

Since this is a quadratic equation, we can determine the maximum b_x-value by using the critical point formula as follows. For the quadratic b_x-profit formula, $a = -2$ and $b = (56 + p_y)$. Thus, the critical point price is

$$p_x = -\frac{b}{2a} = -\frac{56 + p_y}{2 \cdot (-2)} = \frac{1}{4}(56 + p_y).$$

Conclusion: The best reply price to p_y is $\boxed{p_x = \frac{1}{4}(56 + p_y).}$ □

We can now instantly compute the best reply price p_x to any announced price p_y. In particular we can resolve Example 2 in one line as follows.

Example 4: *Suppose that firm Y charges \$22/unit. Use the best reply formula to determine the profit maximizing best reply price for firm X.*
Solution: Substitute $p_y = 22$ into the best reply formula to obtain

$$p_x = \frac{1}{4}(56 + 22) = 19.5$$

□

Exercise 3: *Determine the best reply formula for firm Y by the same method as Example 3.*

The final answer to Exercise 3 is:

$$\boxed{p_y = \tfrac{1}{4}(60 + p_x).}$$

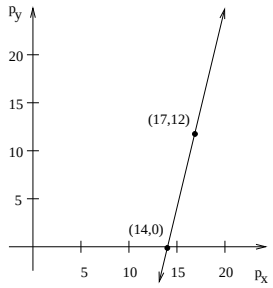
We now use the best reply formulas to find a Nash equilibrium of prices. The graph of a best reply formula is called a *best reply curve*. A Nash equlibrium of prices is given where the best reply curves cross. In this case we are fortunate, because our best reply formulas have linear graphs, and lines are the easiest graphs to plot: Compute two points on the graph and draw the line through them.

Example 5: *Find two points on the graph of* $p_x = \frac{1}{4}(56 + p_y)$ *and use them to plot the graph.*
Solution: Substituting numerical values for p_y into the equation will give numerical values for the corresponding p_x point on the graph.

1. **Substitute** $p_y = 0$**:** $p_x = \frac{1}{4}(56 + 0) = 14$. Therefore $\boxed{(p_x, p_y) = (14, 0)}$ is on the graph of $p_x = \frac{1}{4}(56 + p_y)$.

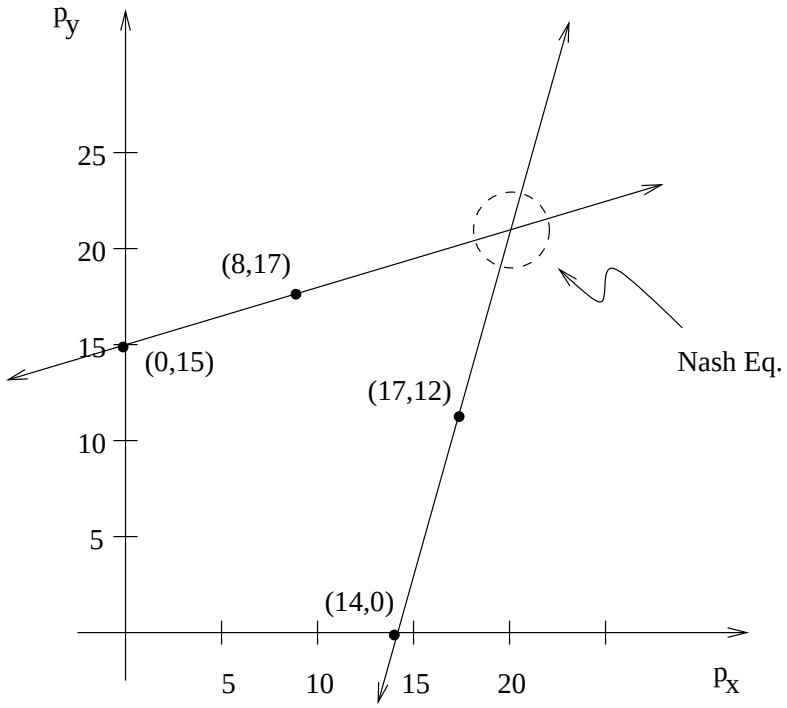
2. **Substitute** $p_y = 12$**:** $p_x = \frac{1}{4}(56{+}12) = 17$. Therefore $\boxed{(p_x, p_y) = (17, 12)}$ is on the graph of $p_x = \frac{1}{4}(56 + p_y)$.

3. **Plot points and draw line:**



□

We may find that $(0, 15)$ and $(8, 17)$ are points on the graph of $p_y = \frac{1}{4}(60 + p_x)$ by the same method. We sketch both best reply curves in the same plot below. Where the curves cross is a Nash equilibrium of prices.



In fact, it is not necessary to graph the best reply curves since there is an algebraic method to determine the coordinates where lines cross.

Example 6: *Find the Nash equilbrium of prices for the best reply formulas* $p_x = \frac{1}{4}(56 + p_y)$ *and* $p_y = \frac{1}{4}(60 + p_x)$.
Solution: Substitute one best reply formula into the other and solve for the price variable.

1. **Substitute the formula for p_y into p_x:**

$$\begin{aligned} p_x &= \tfrac{1}{4}(56 + p_y) \\ &= \tfrac{1}{4}(56 + \tfrac{1}{4}(60 + p_x)) \\ &= 14 + \tfrac{1}{16}(60 + p_x) \\ &= 14 + \tfrac{60}{16} + \tfrac{1}{16}p_x \\ &= \tfrac{71}{4} + \tfrac{1}{16}p_x \end{aligned}$$

2. **Solve for p_x:**

$$\begin{aligned} p_x - \tfrac{1}{16}p_x &= \tfrac{71}{4} \\ \tfrac{15}{16}p_x &= \tfrac{71}{4} \\ p_x &= \tfrac{284}{15} \\ p_x &= 18.9333\ldots \end{aligned}$$

3. **Find p_y by substituting p_x value:**

$$\begin{aligned} p_y &= \tfrac{1}{4}(60 + p_x) \\ &= \tfrac{1}{4}(60 + \tfrac{284}{15}) \\ &= \tfrac{296}{15} \\ &= 19.7333\ldots \end{aligned}$$

Conclusion: A Nash equlibrium of prices is $p_x = \$18.93$ and $p_y = \$19.73$.□

Example 7: *Show that firm X has no incentive to deviate from the Nash equilibrium prices $p_x = 18.93$ and $p_y = 19.73$ by increasing price by 7 cents.*

Solution: First, compute the profits for firm X and Y at the Nash equilibrium. To do this, substitute the values $p_x = 18.93$ and $p_y = 19.73$ into the q_x-quantity formula to obtain

$$q_x = 44 - 2(18.93) + 19.73 = 25.87$$

Then substitute the price and quantity values into the b_x-profit formula

$$b_x = (18.93 - 6)(25.87) = 334.499.$$

Thus, the profit for firm X at the Nash equilibrium of prices is \$334.50.

Second, recompute the q_x-quantity value using the prices $p_x = 19$ and $p_y = 19.73$ to obatin

$$q_x = 44 - 2(19) + 19.73 = 25.73$$

Then substitute the price and quantity values into the b_x=profit formula

$$b_x = (19 - 6)(25.73) = 334.49$$

Thus, the profit for firm X upon raising prices by 7 cents is \$334.49. Hence, firm X has no incentive to raise prices by 7 cents since this makes profit fall by a penny. □

Exercise 7: *Show that firm Y has no incentive to deviate from the Nash equilibrium prices* $p_x = 18.93$ *and* $p_y = 19.73$ *by decreasing price by 73 cents.*