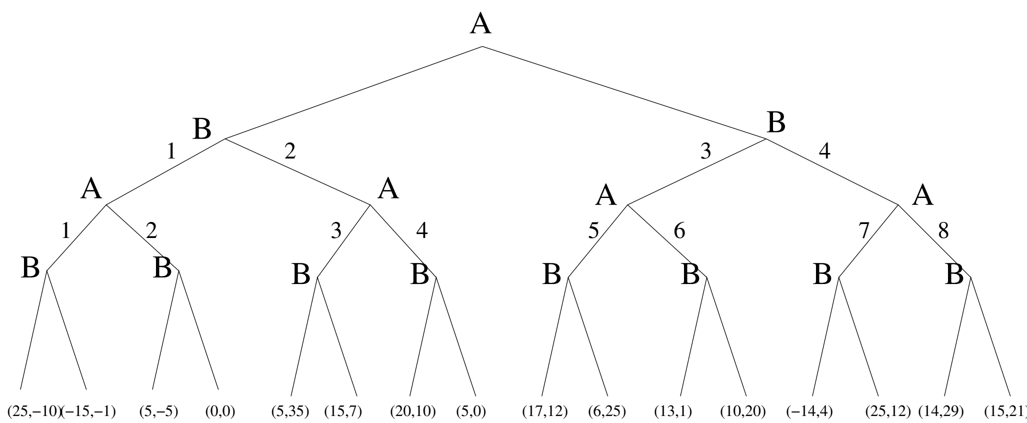
Math 10: The Art and Practice of Mathematics
Worked Examples 3

1. *Consider the game tree below.*



Suppose B can make a single advance commitment regarding their first move followed by A making a single advance commitment regarding their second move.

(a) *Determine the missing payoffs at the terminal nodes of the advance commitment game tree diagrammed below.*

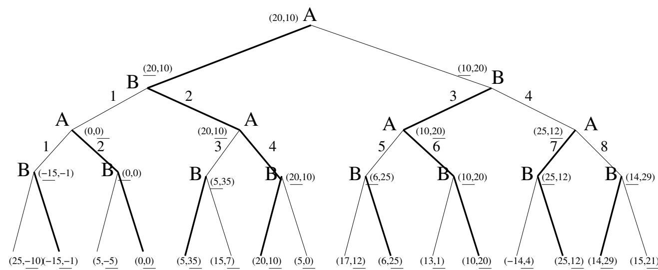
(b) *Find a Nash equilibrium of commitments for the advance commitment game.*

(c) *Classify the commitments you found in part (b) above as threats or promises.*

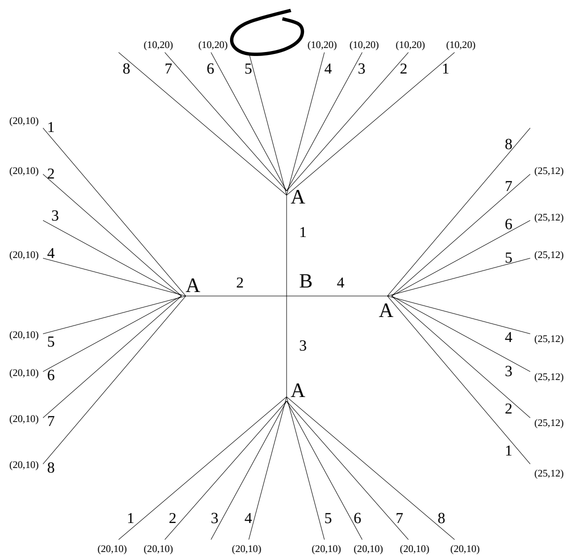
SEE NEXT PAGE FOR SOLUTION

(a) *Determine the missing payoffs at the terminal nodes of the advance commitment game tree diagrammed below.* Proceed as follows:

i. Rollback the game tree at top.

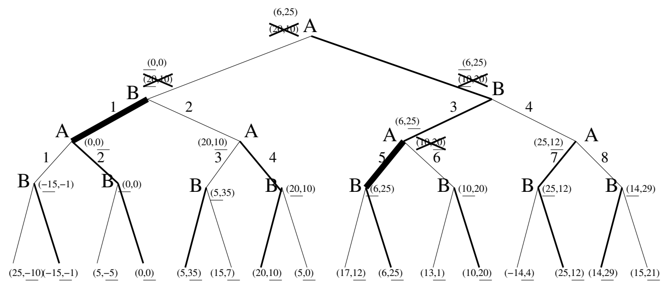


ii. Select the terminal node of the advanced commitment game tree whose payoff you wish to determine. For example,



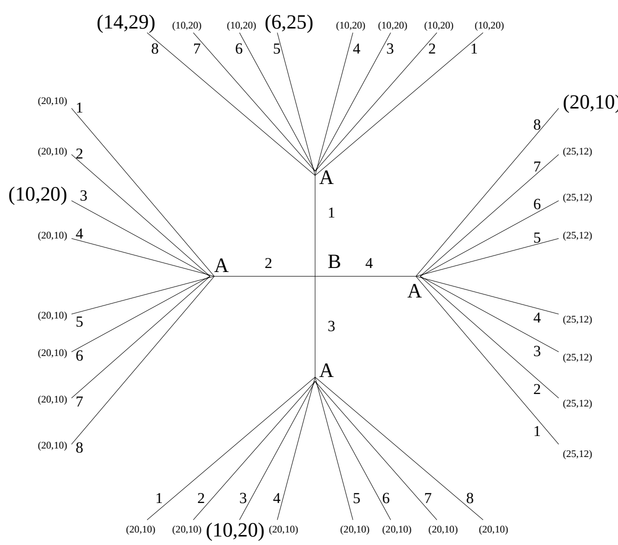
in the diagram above we wish to determine the payoff of making the advance commitments B1 and A5.

iii. Modify the rollback in part i. to reflect the advance commitments B1 and A5 as indicated in the diagram below.



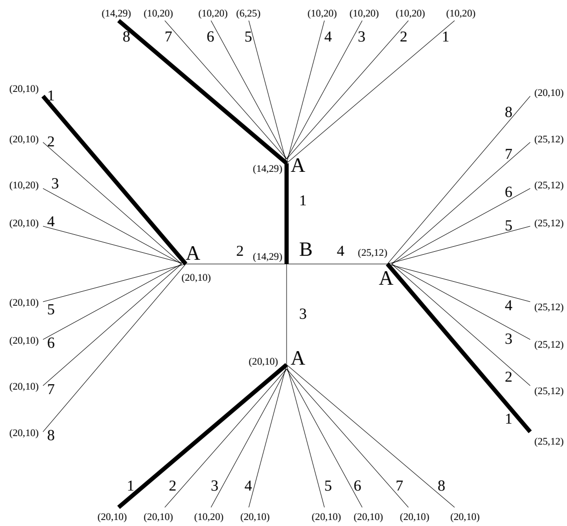
Thus, we enter the payoff (6,25) in the circled terminal node in ii.

Applying this method to determine each missing payoff gives the final answer below.



(b) *Find a Nash equilibrium of commitments for the advance commitment game.*

Rollback the above advance commitment game tree as in the diagram below.



Thus, the rollback commitments are B1 and A8, which yield a rollback payoff of (14,29).

(c) *Classify the commitments you found in part (b) above as threats or promises.*

Below, we compare the rollback payoffs of three scenarios: i. Both A and B uncommitted, ii. B commits to B1, A uncommitted, iii. B commits to B1, A commits to A8.

i. We found in 4(a)i. that, in the absence of advance commitments, the rollback payoff is (20,10).

ii. Using the technique of 4(a)iii., we find that when player B makes advance commitment B1, the rollback payoff becomes (10,20). Since player A's payoff drops from 20 to 10, B1 is termed a *threat*.

iii. We found in 4(a) that commitments B1 and A8 rollback to payoff (14,29). Since player A's advance commitment of A8 caused B's payoff to increase from 20 to 29, A8 is termed a *promise*.

2. *Consider the Stereo Store Cartel Game given by the game table*

Profit in \$100s	D	C
D	(72,72)	(90,54)
C	(54,90)	(81,81)

(a) *Demonstrate that the Stereo Store Cartel Game is a Prisoner's Dilemma.*

First we must show that (D,D) is a Nash equilibrium. This is evident from the best response analysis below.

Profit in \$100s	D	C
D	(72, 72)	(90,54)
C	(54, 90)	(81,81)

Second we must verify that there is the possibility of mutual gain through cooperation. Indeed, (C,C) gives a payoff of (81,81) which both players prefer to the (D,D) payoff of (72,72).

Third we must verify that both players have an incentive to defect from cooperation. Indeed, defecting from cooperation would increase either the row or the column player's score to 90 from 81.

(b) *Suppose that the stores sign a contract promising cooperation (i.e. promising to play row and column C), and that the penalty for deviating (i.e. playing row or column D) is incurring an immediate fine. Determine if the fine resolves the prisoner's dilemma for each of the amounts \$625, \$1,250, and \$1,875.*

Each fine will modify the entries of the game table. The prisoner's dilemma is resolved in the modified game table if (81,81) is a Nash equilibrium payoff.

\$625 fine: In this case, a defecting player has their payoff decreased by 6.25. The modified game table and it's best response analysis can be seen below.

Profit in \$100s	D	C
D	(65.75, 65.75)	(83.75,54)
C	(54, 83.75)	(81,81)

Since (81,81) is not a Nash equilibrium, the \$625 fine does not resolve the prisoner's dilemma.

\$1,250 fine: In this case, a defecting player has their payoff decreased by 12.5. The modified game table and it's best response analysis can be seen below.

Profit in \$100s	D	C
D	(59.5, 59.5)	(77.5,54)
C	(54,77.5)	(81, 81)

Since (81,81) is a Nash equilibrium, the $1,250 fine resolves the prisoner's dilemma.

$1,875 fine: In this case, a defecting player has their payoff decreased by 18.75. The modified game table and it's best response analysis can be seen below.

Profit in $100s	D	C
D	(53.25,53.25)	(71.25, 54)
C	(54 ,71.25)	(81 , 81)

Since (81,81) is a Nash equilibrium, the $1,875 fine resolves the prisoner's dilemma.

3. *Arnold and Bustamante are campaigning for governor. The war-chest of each candidate has enough funds for a single major media advertising blitz.*

   We make the following assumptions:

   (a) *If one wages a successful ad campaign, and the other advertises either not at all or not successfully, then the successful campaign captures 80% of undecided voters and the unsuccessful campaign captures 0%.*

   (b) *If both wage successful ad campaigns, then both capture 40% of the undecided voters.*

   (c) *If both campaign unsuccessfully or not at all, then both capture 0% of the undecided voters.*

   Four weeks before the election, each candidate has a 1/4 chance of a successful ad campaign. Two weeks before the election, each has a 3/4 chance of a successful ad campaign. The night before the election, each has a 100% chance of a successful ad campaign.

   Find a Nash equilibrium of pure-strategies for the gubernatorial campaign specifying in each stage whether the candidate advertises or waits.

   We begin with best-reply analysis for the night before the election stage game table. In the table below W=wait, A=advertise.

Night before	W	A
W	(0,0)	(0, 80)
A	(80 ,0)	(40 , 40)

Thus (A,A) is the Nash equilibrium for the night before stage, with a payoff of $(40, 40)$.

Next, find the payoffs to fill out the entries of the two weeks stage as follows.

Payoff of (W,W) is (40,40): To see this, notice that given the hypothetical play (W,W), both candidates advance to the night before the election stage game, where the payoff is $(40,40)$.

Payoff of (A,W) is (60,20): To see this, apply the expected payoff formula to the two possible outcomes:

$$\begin{aligned} \text{exp.pay.} &= \text{prob(A succ)} \cdot \text{pay(A succ)} + \text{prob(A fails)} \cdot \text{pay(A fails)} \\ &= (3/4)(80,0) + (1/4)(0,80) \\ &= (60,20). \end{aligned}$$

Payoff of (W,A) is (20,60): This is very similar to the computation above, but with the roles of A and B reversed.

Payoff of (A,A) is (37.5,37.5): To see this, apply the expected payoff formula to the four possible outcomes:

$$\begin{aligned} \text{exp.pay.} &= \text{prob(A succ,B succ)} \cdot \text{pay(A succ, B succ)} \\ &\quad + \text{prob(A succ,B fail)} \cdot \text{pay(A succ, B fail)} \\ &\quad + \text{prob(A fail,B succ)} \cdot \text{pay(A fail, B succ)} \\ &\quad + \text{prob(A fail,B fail)} \cdot \text{pay(A fail, B fail)} \\ &= (3/4)(3/4)(40,40) + (3/4)(1/4)(80,0) \\ &\quad + (1/4)(3/4)(0,80) + (1/4)(1/4)(0,0) \\ &= (37.5,37.5) \end{aligned}$$

We now perform a best-reply analysis for the two weeks before the election stage game table. In the table below W=wait, A=advertise.

2 Weeks	W	A
W	(40,40)	(20, [60])
A	([60],20)	([37.5], [37.5])

Thus, (A,A) is the Nash equilibrium for the two weeks stage, with a payoff of $(37.5,37.5)$.

Next, find the payoffs of the four weeks stage as follows.

Payoff of (W,W) is (37.5,37.5): To see this, notice that given the hypothetical play (W,W), both players advance to the two weeks stage, where the expected payoff is $(37.5,37.5)$.

To compute the other payoffs, notice that the entire computation is the same as in the two weeks stage, except with the probabiities are reversed: prob(succ) changes from 3/4 to 1/4, and prob(fail) changes from 1/4 to 3/4. Thus,

Payoff of (A,W) is (20,60): To see this, apply the expected payoff formula to the two possible outcomes:

$$\begin{aligned} \text{exp.pay.} &= \text{prob(A succ)} \cdot \text{pay(A succ)} + \text{prob(A fails)} \cdot \text{pay(A fails)} \\ &= (1/4)(80,0) + (3/4)(0,80) \\ &= (20,60). \end{aligned}$$

Payoff of (W,A) is (60,20): This is very similar to the computation above, but with the roles of A and B reversed.

Payoff of (A,A) is (17.5,17.5): To see this, apply the expected payoff formula to the four possible outcomes:

$$\begin{aligned} \text{exp.pay.} &= \text{prob(A succ,B succ)} \cdot \text{pay(A succ, B succ)} \\ &\quad + \text{prob(A succ,B fail)} \cdot \text{pay(A succ, B fail)} \\ &\quad + \text{prob(A fail,B succ)} \cdot \text{pay(A fail, B succ)} \\ &\quad + \text{prob(A fail,B fail)} \cdot \text{pay(A fail, B fail)} \\ &= (1/4)(1/4)(40,40) + (1/4)(3/4)(80,0) \\ &\quad + (3/4)(1/4)(0,80) + (3/4)(3/4)(0,0) \\ &= (17.5,17.5) \end{aligned}$$

We now perform a best-reply analysis for the four weeks stage. In the table below W=wait, A=advertise.

4 Weeks	W	A
W	(\[37.5\],\[37.5\])	(\[60\],20)
A	(20,\[60\])	(17.5,17.5)

Thus, (W,W) is the Nash equilibrium for the four weeks stage game, with a payoff of $(37.5, 37.5)$.

The Nash equilbrium of strategies is for both candidates to advertise two weeks before the election.

4. *Suppose that Mobilex and Chevtex are two gas stations that can charge either Low or High prices for gas, and that their monthly profits are given by the Gas Price Game table below.*

Profit in \$1000s	Low	High
Low	(246,246)	(253,243)
High	(243,253)	(250,250)

Notice that the Gas Price Game is a Prisoner's Dilemma. Suppose that the interest rate is 4% and that Chevtex in on the verge of bankruptcy; in fact, each month the chance that Chevtex will shutter is 4/7. Assuming that Chevtex follows the Tit-for-Tat price strategy, determine which of the following Mobilex price strategies is the best: (a) Never defect, (b) Defect only once, (c) Defect forever.

This is a game of indefinite repetition with interest rate $r = .04$ and probability of repetition $p = 1 - 4/7 = 3/7$. We will approach this by solving the infinitely repeated Gas Price Game subject to the risk adjusted interest rate R. The risk adjusted interest rate formula gives

$$R = \frac{1+r}{p} - 1 = \frac{1+.04}{3/7} - 1 = 1.42666\ldots,$$

or about 143%.

We now solve the infinitely repeated Gas Price Game with interest rate 143%. Let Mobilex be row player A, and let Chevtex be column player B. We first find the sequence of moves and payoffs that will result from each of the three given Mobilex price strategies (a)–(c):

Mobilex price strategy (a):

Stage	1	2	3	4	...
A's Move	C	C	C	C	...
B's Move	C	C	C	C	...
A's pay	250	250	250	250	...

Mobilex price strategy (b):

Stage	1	2	3	4	...
A's Move	D	C	C	C	...
B's Move	C	D	C	C	...
A's pay	253	243	250	250	...

Mobilex price strategy (c):

Stage	1	2	3	4	...
A's Move	D	D	D	D	...
B's Move	C	D	D	D	...
A's pay	253	246	246	246	...

Let us determine the present value of the total gain (or loss) of the defect once strategy (b) over the never defect strategy (a). As we can see from the tables above, the only difference in payoff occur in stages 1 and 2. Notice that the gain in stage 1 is 253-250 = 3 and that the gain in stage 2 is 243-250=-7 (which is a loss of 7). Thus, the present value of the total gain (or loss) is

$$3 + \frac{-7}{1+1.43} = .119\ldots$$

where the fraction above is gotten by substitution of numeric values into the present value formula $PV = \frac{A}{1+R}$. This shows that the defect once strategy is better than the never defect strategy. In particular, defecting once yields a gain of present value $119.

Let us determine the present value of the total gain (or loss) of the always defect strategy (c) over the never defect strategy (a). We look at the difference in payoffs one stage at a time, and organize our results in the table below:

Gain (or loss):

Stage	1	2	3	4	...
Gain (or loss)	3	-4	-4	-4	...
PV	3	$\frac{-4}{1+1.43}$	$\frac{-4}{(1+1.43)^2}$	$\frac{-4}{(1+.43)^3}$	...

We add up the present value of the gain (or loss) in each stage using the geometric series formula:

$$\begin{aligned} 3 + \tfrac{-4}{2.43} + \tfrac{-4}{(2.43)^2} + \tfrac{-4}{(2.43)^3} + \cdots &= 3 - \left(\tfrac{4}{2.43}\right)\left[1 + \left(\tfrac{1}{2.43}\right) + \left(\tfrac{1}{2.43}\right)^2 + \cdots\right] \\ &= 3 - \left(\tfrac{4}{2.43}\right)\left[\tfrac{1}{1-\frac{1}{2.43}}\right] \text{ (geom.series.formula)} \\ &= 3 - \tfrac{4}{2.43-1} \\ &= .203 \end{aligned}$$

This shows that the always defect strategy (c) is better than the never defect strategy (a). In particular, defecting always yields a gain of present value \$203.

Conclusion: The best price strategy for Mobilex is to always defect.