Math 10: The Art and Practice of Mathematics
Worked Examples 2

1. *Determine maximin mixed-strategies producing a Nash equilibrium for the tennis-shot game given by the table below.*

Success %	DL	CC
DL	(40,60)	(80,20)
CC	(90,10)	(20,80)

Determine Player A's maximin strategy as follows. Suppose A plays DL with prob p, and CC with prob $1-p$.

B defends DL: Apply the expected payoff formula to the two possible outcomes;

$$\begin{aligned} E_A(\text{B def DL}) &= \text{prob(A pl DL)pay}_A\text{(A pl DL)} \\ &\quad +\text{prob(A pl CC)pay}_A\text{(A pl CC)} \\ &= (p)(40)+(1-p)(90) \\ &= 90-50p \end{aligned}$$

Thus, when B defends DL, A's expected payoff is $90-50p$.

B defends CC: Apply the expected payoff formula to the two possible outcomes;

$$\begin{aligned} E_A(\text{B def CC}) &= \text{prob(A pl DL)pay}_A\text{(A pl DL)} \\ &\quad +\text{prob(A pl CC)pay}_A\text{(A pl CC)} \\ &= (p)(80)+(1-p)(20) \\ &= 20+60p \end{aligned}$$

Thus, when B defends CC, A's expected payoff is $20+60p$.

A's maximin strategy is the crossover value p^* making these payoff formulas equal.

$$\begin{aligned} 90-50p^* &= 20+60p^* \\ 70 &= 110p^* \\ p^* &= 70/110 \end{aligned}$$

Thus, A's maximin strategy is to play DL with probability 70/110, which is approximately 63.6% of the time.

Determine Player B's maximin strategy as follows. Suppose B defends DL with prob q, and CC with prob $1-q$.

A plays DL: Apply the expected payoff formula to the two possible outcomes;

$$\begin{aligned} E_B(\text{A pl DL}) &= \text{prob(B def DL)pay}_B\text{(B def DL)} \\ &\quad + \text{prob(B def CC)pay}_B\text{(B def CC)} \\ &= (q)(60) + (1-q)(20) \\ &= 20 + 40q \end{aligned}$$

Thus when A plays DL, B's expected payoff is $20 + 40q$.

A plays CC: Apply the expected payoff formula to the two possible outcomes;

$$\begin{aligned} E_B(\text{A pl CC}) &= \text{prob(B def DL)pay}_B\text{(B def DL)} \\ &\quad + \text{prob(B def CC)pay}_B\text{(B def CC)} \\ &= (q)(10) + (1-q)(80) \\ &= 80 - 70q \end{aligned}$$

Thus when A plays CC, B's expected payoff is $80 - 70q$.

B's maximin strategy is the crossover value q^* making these payoff formulas equal.

$$\begin{aligned} 20 + 40q^* &= 80 - 70q^* \\ 60 &= 110q^* \\ q^* &= 60/110 \end{aligned}$$

Thus, B's maximin strategy is to play DL with probability 60/110, which is approximately 54.5% of the time.

2. *Use best-response analysis to find all Nash equilibria of mixed-strategies for the battle-of-the-buddies game given by the table below.*

Satisfaction	Starbucks	Peet's
Starbucks	(3,1)	(0,0)
Peet's	(0,0)	(1,3)

Determine Player B's best reply rule as follows. Suppose A plays SB with prob p, and PT with prob $1-p$.

B plays SB: Apply the expected payoff formula to the two possible outcomes;

$$\begin{aligned} E_B(\text{B pl SB}) &= \text{prob(A pl SB)pay}_B\text{(A pl SB)} \\ &\quad +\text{prob(A pl PT)pay}_B\text{(A pl PT)} \\ &= (p)(1) + (1-p)(0) \\ &= p \end{aligned}$$

Thus when B plays SB, B's expected payoff is p.

B plays PT: Apply the expected payoff formula to the two possible outcomes;

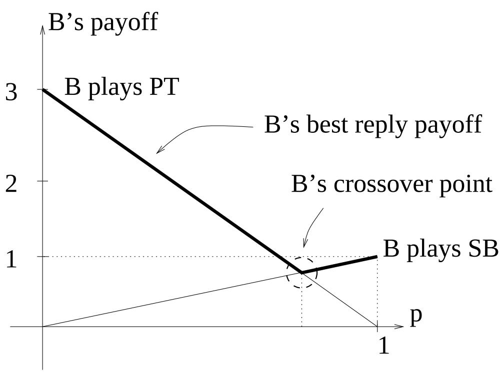
$$\begin{aligned} E_B(\text{B pl PT}) &= \text{prob(A pl SB)pay}_B\text{(A pl SB)} \\ &\quad +\text{prob(A pl PT)pay}_B\text{(A pl PT)} \\ &= (p)(0) + (1-p)(3) \\ &= 3-3p \end{aligned}$$

Thus when B plays PT, B's expected payoff is $3-3p$.

The graph of B's payoff in each case appears on the top of the next page.

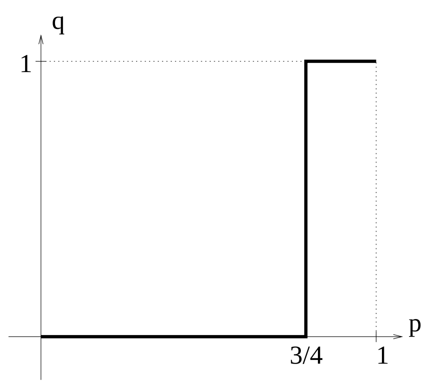
B's crossover point is given by p^* making these payoff formulas equal.

$$\begin{aligned} p^* &= 3-3p^* \\ 4p^* &= 3 \\ p^* &= 3/4 \end{aligned}$$



Thus, B's best reply is to play SB when $p \geq 3/4$, and to play PT when $p \leq 3/4$.

To graph B's best reply curve, we let q denote the probability that B will play SB; i.e. B always plays SB when $q = 1$, and B always plays PT when $q = 0$. With this convention, B's best reply curve is graphed below.



Determine Player A's best reply rule as follows. Suppose B plays SB with prob q, and PT with prob $1 - q$.

A plays SB: Apply the expected payoff formula to the two possible outcomes;

$$\begin{aligned} E_A(\text{A pl SB}) &= \text{prob(B pl SB)pay}_A\text{(B pl SB)} \\ &\quad + \text{prob(B pl PT)pay}_A\text{(B pl PT)} \end{aligned}$$

$$
\begin{aligned}
&= (q)(3) + (1-q)(0) \\
&= 3q
\end{aligned}
$$

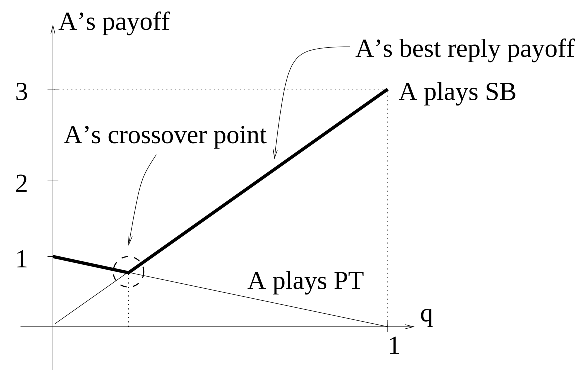
Thus, when A plays SB, A's expected payoff is $3q$.

A plays PT: Apply the expected payoff formula to the two possible outcomes;

$$
\begin{aligned}
E_A(\text{A pl PT}) &= \text{prob(B pl SB)pay}_A\text{(B pl SB)} \\
&\quad +\text{prob(B pl PT)pay}_A\text{(B pl PT)} \\
&= (q)(0) + (1-q)(1) \\
&= 1-q
\end{aligned}
$$

Thus, when A plays PT, A's expected payoff is $1-q$.

The graph of A's payoff in each case appears below.

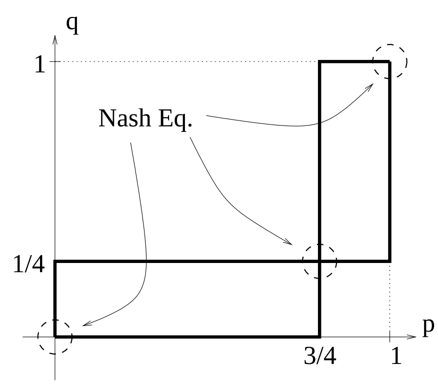


A's crossover point is given by q^*making these payoff formulas equal.

$$
\begin{aligned}
3q^* &= 1-q^* \\
4q^* &= 1 \\
q^* &= 1/4
\end{aligned}
$$

Thus, A's best reply is to play SB when $q \geq 1/4$, and to play PT when $q \leq 1/4$.

To graph A's best reply curve, recall that p denotes the probability that A will play SB; i.e. A always plays SB when $p = 1$, and A always plays PT when $p = 0$. With this convention, A's best reply curve is juxtaposed on B's best reply curve below. The Nash equilibria are

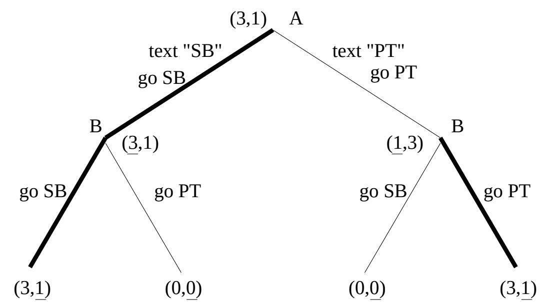


found where the best reply curves cross. In this case, there are three crossing points: $(p, q) = (0, 0)$ which corresponds to the pure strategies (PT,PT), $(p, q) = (SB, SB)$ which corresponds to the pure strategies (SB,SB), and $(p, q) = (3/4, 1/4)$, which is a pair of mixed strategies.

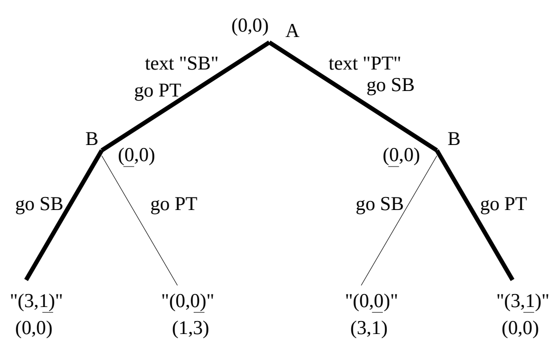
3. *What is the strategic effect of allowing the row player to send a single text message to the column player prior to game play in the* Battle of the Buddies *given in problem 2?*

We will determine the (T,B) and (F,B) payoffs in the communication stage of the one message game for *Battle of the Buddies*.

(T,B): Player A sends a true text message which is believed by player B. We have sketched and rolled back the relevant game tree below. This shows that the rollback strategies are for player A to send the true text message "SB," and for player B to go to SB. The rollback payoff is $(3, 1)$.



(F,B): Player A sends a false text message which is believed by player B. We have sketched and rolled back the relevant game tree below. This shows that the rollback strategies are for player A to send either false text message, "SB" or "PT," and for player B to go to the destination of the text message. In either case, the rollback payoff is $(0, 0)$.



Since player A has no incentive to send a false message, the players select the (T,B) strategies by the principle of direct communication. Thus, the strategic effect of communication is to guarantee that the players meet up at Starbucks for a payoff of (3,1).

4. *Suppose motels come in two types: clean and smelly. Both types of motels can get annual AAA certification, but for the smelly motels certification is more costly to obtain. Suppose that clean motels have to spend C dollars annually preparing for certification; smelly motels have to spend $3C$ dollars annually preparing for certification. AAA certified motels can earn $2,000,000 annually. Motels without AAA certification earn only $1,250,000 annually. What is the range of C values for which a clean motel will chose to signal with this device but a smelly motel will not?*

   Consider first the smelly motels. For a smelly motel, the benefit of AAA certification is $2,000,000, but the cost of obtaining certification is $3C$. Thus, a smelly motel with certification realizes a payoff of $2,000,000 - 3C$. On the other hand, an uncertified smelly motel realizes a payoff of $1,250,000$. Thus, for the certification requirements to be stringent enough to screen out smelly motels, AAA must set C large enough so that

$$2,000,000 - 3C < 1,250,000.$$

   Adding $3C$ and subtracting $1,250,000$ from both sides gives $750,000 < 3C$. Dividing both sides by 3 gives $250,000 < C$.

   Consider next the clean motels. For a clean motel, the benefit of AAA certification is $2,000,000, but the cost of obtaining certification is C. Thus, a clean motel with certification realizes a payoff of $2,000,000 - C$. On the other hand, an uncertified clean motel realizes a payoff of $1,250,000$. Thus, for the certification requirements to be lenient enough to admit clean motels, AAA must set C small enough so that

$$2,000,000 - C > 1,250,000.$$

   Adding C and subtracting $1,250,000$ to both sides gives $750,000 > C$.

   Conclusion: For any C value between $250,000 and $750,000, clean motels will chose to signal with AAA certification, and smelly motels will not.

5. *Suppose that IBM has an IT problem which, if solved, will increase IBM's profits by $2,400,000. IBM wishes to outsource solving the IT problem to Google. Google charges $400,000 for standard service, for which Google makes a profit of $40,000. Google charges $600,000 for intensive service, for which Google makes a profit of $80,000. Notice that if Google charges for intensive service, but provides only standard service, then Google makes a profit of $240,000. In a project proposal to IBM, Google estimates that standard service has a 60% chance of solving the IT problem, and intensive service has an 80% chance of solving the IT problem.*

   (a) *Show that a principal-agent problem arises if IBM hires Google and pays a flat fee up-front for service.*

   We begin by computing the entries of the Flat Fee Game table below.

Profit in $1000s	Std Svc	Int Svc if Paid
Pay Std Fee	?	?
Pay Int Fee	?	?

   Here IBM is the row player and Google is the column player.

   Computing the (Std,Std) payoff: Google makes a guaranteed profit of $40,000, which is represented by the payoff 40. IBM's payoff may be determined by the following expected value calculation:

$$\begin{aligned} \text{exp.pay.} &= \text{prob(solved)pay(solved)+prob(not solved)pay(not solved)} \\ &= (.60)(\$2,400,000 - \$400,000) + (.40)(-\$400,000) \\ &= \$1,040,000. \end{aligned}$$

   This is represented by the payoff 1040. Thus, the (Std,Std) payoff is (1040,40).

   Computing the (Std,Int) payoff: Recall that Google only provides intensive service if they are paid to do so, and in this case, IBM is paying only for standard service. Thus, the (Std,Int) payoff is the same as the (Std,Std) payoff: (1040,40)

   Computing the (Int,Int) payoff: Google makes a guaranteed profit of $80,000, which is represented by the payoff 80. IBM's payoff may be determined by the following expected value calculation:

$$\begin{aligned} \text{exp.pay.} &= \text{prob(solved)pay(solved)+prob(not solved)pay(not solved)} \\ &= (.80)(\$2,400,000 - \$600,000) + (.20)(-\$600,000) \\ &= \$1,320,000. \end{aligned}$$

This is represented by the payoff 1320. Thus, the (Int,Int) payoff is (1320,80).

Computing the (Int,Std) payoff: Google makes a guaranteed profit of $240,000, which is represented by the payoff 240. IBM's payoff may be determined by the following expected value calculation:

$$\begin{aligned} \text{exp.pay.} &= \text{prob(solved)pay(solved)+prob(not solved)pay(not solved)} \\ &= (.60)(\$2,400,000 - \$600,000) + (.40)(-\$600,000) \\ &= \$840,000. \end{aligned}$$

This is represented by the payoff 840. Thus, the (Int,Std) payoff is (840,240).

Thus the Flat Fee Game table is given below.

Profit in $1000s	Std Svc	Int Svc if Paid
Pay Std Fee	(1040,40)	(1040,40)
Pay Int Fee	(840,240)	(1320,80)

We will now show that this Flat Fee Game table has a principal-agent problem wherein IBM is the principal and Google is the agent. To do this we must analyze the game table for the one-message game. We begin by computing the relevant entries of the One-Message game table.

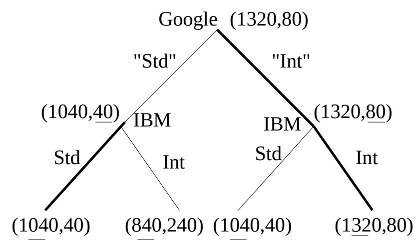
Profit in $1000s	Believe	Disregard
True	?	X
False	?	?

Computing the (F,D) payoff: We simply find the Nash equilibrium payoff of the Flat Fee Game table:

Profit in $1000s	Std Svc	Int Svc if Paid
Pay Std Fee	([1040],[40])	(1040,[40])
Pay Int Fee	(840,[240])	([1320],80)

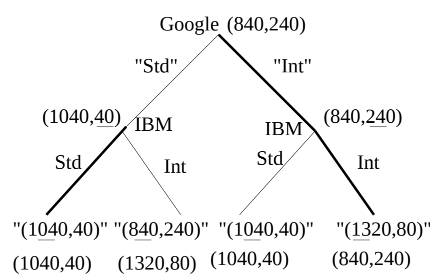
Hence, (1040,40) is the Nash equilibrium payoff of the Flat Fee Game. Since Google is the agent (message-sender) and IBM is the principal (message-receiver), these payoffs should be entered into the (F,D) entry of the one-message game table as (40,1040).

Computing the (T,B) payoff: We rollback the game tree for the One Message Game where Google sends a True fax (either "Standard" or "Intensive") which is believed by IBM. This is illustrated below.



The rollback payoff is (1320,80).

Computing the (F,B) payoff: We rollback the game tree for the One Message Game where Google sends a False fax (either "Standard" or "Intensive") which is believed by IBM. This is illustrated below.



The rollback payoff is (840,240).

The rollback payoffs must be entered into the One Message Game table in reversed form because Google is the column player of the original game table but is the message sender who is the row player of the One Message Game. Thus, the One Message Game table is

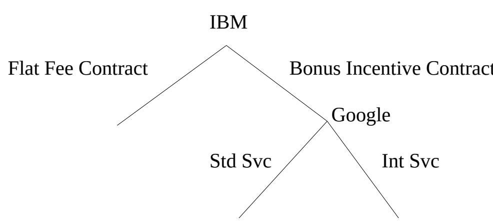
Profit in $1000s	Believe	Disregard
True	(80,1320)	X
False	(240,840)	(40,1040)

There is an *opportunity for mutual gain through cooperation* since both players prefer the (T,B) payoff of (80,1320) to the (F,D) payoff of (40,1040). There is also *moral hazard* for the agent since the agent has an incentive to lie, increasing payoff from 80 to 240.

Conclusion: . There is a principal-agent problem.

(b) *Consider the bonus-incentive contract wherein Google provides up-front $200,000 of noncompensated work to IBM and earns a $1,000,000 bonus if and only if the IT problem is solved. Does this bonus-incentive contract resolve the principal-agent problem in part (a)?*

Yes, it does. A resolution of the principal-agent problem is a strategic move which creates a Nash equilibrium of strategies yielding the same payoff as that of (T,B) in the One Message Game. Offering this bonus-incentive contract is such a strategic move on the part of IBM. To see this, consider the Contract Offer Game tree below. We will determine the payoff at each terminal node of



this game, and then rollback the game tree to show that the Nash equilibrium of strategies gives a payoff of (1320,80).

Flat Fee Contract: We have already shown about that the Flat Fee Game table has Nash equilibrium (1040,40).

Bonus-Incentive Contract and Standard Service: We determine IBM's profit with the following expected value calculation:

$$\begin{aligned} \text{exp.pay.} &= \text{prob(solved)pay(solved)+prob(not solved)pay(not solved)} \\ &= (.60)(\$2,400,000 + \$200,000 - \$1,000,000) + (.40)(\$200,000) \\ &= \$1,040,000. \end{aligned}$$

We determine Google's profit with a similar expected value calculation:

$$\begin{aligned}\text{exp.pay.} &= \text{prob(solved)pay(solved)+prob(not solved)pay(not solved)}\\ &= (.60)(-\$200,000 - \$360,000 + \$1,000,000)\\ &\quad +(.40)(-\$360,000 - \$200,000)\\ &= \$40,000\end{aligned}$$

Remark: The -$360,000 figure in the computation above is Google's overhead cost on providing standard service; i.e. Google's fee - Google's profit = $400,000 - $40,000= $360,000.

IBM and Google's profits are then represented by the payoff (1040,40).

Bonus-Incentive Contract and Intensive Service: We determine IBM's profit with the following expected value calculation:

$$\begin{aligned}\text{exp.pay.} &= \text{prob(solved)pay(solved)+prob(not solved)pay(not solved)}\\ &= (.80)(\$2,400,000 + \$200,000 - \$1,000,000) + (.20)(\$200,000)\\ &= \$1,320,000.\end{aligned}$$

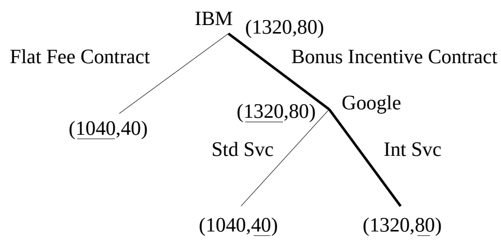
We determine Google's profit with a similar expected value calculation:

$$\begin{aligned}\text{exp.pay.} &= \text{prob(solved)pay(solved)+prob(not solved)pay(not solved)}\\ &= (.80)(-\$200,000 - \$520,000 + \$1,000,000)\\ &\quad +(.20)(-\$200,000 - \$520,000)\\ &= \$80,000\end{aligned}$$

Remark: The -$520,000 figure in the computation above is Google's overhead cost on providing intensive service; i.e. Google's fee - Google's profit = $600,000 - $80,000= $520,000.

IBM and Google's profits are then represented by the payoff (1320,80).

Finally, we write these payoffs into the Contract Offer Game tree and rollback as below to obtain the Nash equilibrium payoff (1320,80).



Conclusion: The Bonus Incentive Contract solves the Principal-Agent problem.